MOTIVIC ZETA FUNCTIONS OF ABELIAN VARIETIES, AND THE MONODROMY CONJECTURE

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ABSTRACT. We prove for abelian varieties a global form of Denef and Loeser's motivic monodromy conjecture, in arbitrary characteristic. More precisely, we prove that for every tamely ramified abelian variety A over a complete discretely valued field, its motivic zeta function has a unique pole at Chai's base change conductor $c(A)$ of A, and that the order of this pole equals one plus the potential toric rank of A. Moreover, we show that for every embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, the value $\exp(2\pi i c(A))$ is an ℓ-adic tame monodromy eigenvalue of A. The main tool in the paper is Edixhoven's filtration on the special fiber of the Néron model of A, which measures the behaviour of the Néron model under tame base change.

1. INTRODUCTION

Let K be a complete discretely valued field with ring of integers R and algebraically closed residue field k. We denote by p the characteristic exponent of k, and by $\mathbb{N}'$ the set of strictly positive integers that are prime to p. We fix a prime $\ell \neq p$.

Given a smooth, proper and connected K-variety X and a gauge form ω on X (i.e., a nowhere vanishing differential form of maximal degree), one can associate to the pair (X, ω) a *motivic generating series* $S(X, \omega; T)$ as follows. For each $d \in \mathbb{N}'$, the field K admits an extension $K(d)$ of degree d which is unique up to K-isomorphism. If we put $X(d) = X \times_K K(d)$ and if we denote by $\omega(d)$ the pull-back of ω to $X(d)$, then the motivic generating series $S(X, \omega; T)$ is given by

$$S(X, \omega; T) = \sum_{d \in \mathbb{N}'} \left(\int_{X(d)} |\omega(d)| \right) T^d \in \mathcal{M}_k[[T]]$$

Here $\mathcal{M}_k$ denotes the localized Grothendieck ring of k-varieties, and

$$\int_{X(d)} |\omega(d)| \in \mathcal{M}_k$$

is the motivic integral of the gauge form $\omega(d)$ on $X(d)$. These motivic integrals were defined in [22] and can be computed on a weak Néron model of $X(d)$ (see Proposition 2.3).

The aim of this paper is to study the series $S(A, \omega; T)$ when A is a tamely ramified abelian K-variety and ω is a "normalized Haar measure" on A, i.e., a translation-invariant gauge form that extends to a relative gauge form on the Néron model of A.

Such a normalized Haar measure always exists, and it is unique up to multiplication with a unit in R. It follows that $S(A, \omega; T)$ only depends on A. We denote the series $\mathbb{L}^{\dim(A)} \cdot S(A, \omega; T)$ by $Z_A(T)$, and we call it the *motivic zeta function* of the abelian K-variety A[1].

It is easily seen that

$$(1.1) \qquad Z_A(T) = \sum_{d \in \mathbb{N}'} [\mathcal{A}(d)_s] \mathbb{L}^{-ord_{\mathcal{A}(d)_s^o}(\omega(d))} T^d \quad \in \mathcal{M}_k[[T]]$$

where $\mathcal{A}(d)$ is the Néron model of $A(d)$, and $ord_{\mathcal{A}(d)_s^o}(\omega(d))$ denotes the order of the gauge form $\omega(d)$ on $A(d)$ along the identity component $\mathcal{A}(d)_s^o$ of the special fiber of $\mathcal{A}(d)$. So the zeta function $Z_A(T)$ depends on two (related) factors: the behaviour of the Néron model of A under tame base change, and the function $d \mapsto ord_{\mathcal{A}(d)_s^o}(\omega(d))$.

Our analysis of both factors heavily relies upon Edixhoven's results in [13]. He constructs a filtration on the special fiber $\mathcal{A}_s$ of the Néron model of A, which measures the behaviour of the Néron model under tame base change. We prove that $ord_{\mathcal{A}(d)_s^o}(\omega(d))$ can be expressed in terms of the jumps in this filtration. We also prove that the sum of the jumps is equal to the base change conductor $c(A)$ introduced by Chai in [8].

In [13], Edixhoven shows as well that the Néron model of A is canonically isomorphic to the $G(K(d)/K)$-invariant part of the Weil restriction of the Néron model of $A(d)$. This result allows us to analyze the behaviour of $[\mathcal{A}(d)_s] \in \mathcal{M}_k$ when d varies. If we denote by $\phi_A(d)$ the number of connected components of $\mathcal{A}(d)_s$, then we have $[\mathcal{A}(d)_s] = \phi_A(d)[\mathcal{A}(d)_s^o]$. We show that $[\mathcal{A}(d)_s^o]$ only depends on d modulo e, with e the degree of the minimal extension of K where A acquires semi-abelian reduction. The behaviour of $\phi_A(\cdot)$ is interesting in its own right; we studied it in the independent paper [16].

The first question that arises is the rationality of $Z_A(T)$, and the nature of its poles. We prove that $Z_A(T)$ is rational, and belongs to the subring

$$\mathcal{M}_k \left[T, \frac{1}{1 - \mathbb{L}^a T^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}, \ a/b = c(A)}$$

of $\mathcal{M}_k[[T]]$. In particular, the series $Z_A(\mathbb{L}^{-s})$ has a unique pole at $s = c(A)$. We show that the order of this pole equals one plus the potential toric rank of A. Our proof does not use resolution of singularities.

A second question we consider is the relation between the pole $s = c(A)$ and the monodromy eigenvalues of A. To explain the motivation behind this question, we need some background. Let X be a smooth connected complex variety, endowed with a dominant morphism $f : X \to \operatorname{Spec} \mathbb{C}[t]$. Let x be a closed point of the special fiber of f. We denote by $\mathscr{F}_x$ the analytic Milnor fiber of f at x [29, §9.1]. It is a separated smooth rigid $\mathbb{C}((t))$-variety, and it serves as a non-archimedean model for the topological Milnor fibration of f at x. If ϕ/dt is a so-called *Gelfand-Leray form* on $\mathscr{F}_x$ [29, 9.5] then the series $S(\mathscr{F}_x, \phi/dt; T) \in \mathcal{M}_\mathbb{C}[[T]]$ can be defined in a similar way as above. Up to normalization, it coincides with Denef and Loeser's motivic zeta function of f at x [26, 9.7]. Denef and Loeser's monodromy conjecture predicts that, if $\alpha \in \mathbb{Q}$ is a pole of $S(\mathscr{F}_x, \phi/dt; \mathbb{L}^{-s})$, then $\exp(2\pi i \alpha)$ is a monodromy

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[1]This notion should not be confused with Kapranov's motivic zeta function of A, which is studied, for instance, in [17]. The two are not related in any direct way.

eigenvalue of f at some closed point of the special fiber of f. The conjecture has been solved, for instance, in the case where X is a surface [21][33], but the general case remains wide open.

We will formulate a global form of this conjecture, and prove it for abelian varieties. Denote by $\tau(c(A))$ the order of $c(A)$ in the group $\mathbb{Q}/\mathbb{Z}$, and by $\Phi_{\tau(c(A))}(t)$ the cyclotomic polynomial whose roots are the primitive roots of unity of order $\tau(c(A))$. Let σ be a topological generator of the tame monodromy group $G(K^t/K)$. We give a formula for the characteristic polynomial of the action of σ on the Tate module of A, in terms of the jumps in Edixhoven's filtration. We also prove that $\Phi_{\tau(c(A))}(t)$ divides the characteristic polynomial of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$, with g the dimension of A. Since $s = c(A)$ is the only pole of $Z_A(\mathbb{L}^{-s})$, this yields a global form of Denef and Loeser's monodromy conjecture for abelian varieties.

Of course, we hope that our proofs and results will lead to new insights into the local case of the conjecture. In this respect, it is intriguing that the Greenberg schemes appearing in the construction of Edixhoven's filtration also play a fundamental role in the theory of motivic integration and local zeta functions [29].

Although the main results in this paper concern abelian varieties, we develop a large part of the theory in greater generality, in order to include the case of semi-abelian varieties. It would be interesting to know if the monodromy conjecture holds also for tamely ramified semi-abelian varieties. The case of algebraic tori is treated in the follow-up paper [28]. The assumption that A is tamely ramified is crucial for our arguments; it is a challenging problem to adapt our results to wildly ramified (semi-)abelian varieties.

To conclude this introduction, we give a survey of the structure of the paper. Section 2 gathers some preliminaries on motivic integration and motivic generating series. In Definition 2.6, we formulate a global version of Denef and Loeser's monodromy conjecture.

Section 3 deals with Néron models of smooth commutative algebraic K-groups G. The Néron models we consider are the maximal quasi-compact open subgroups of the locally of finite type Néron model $\mathcal{G}^{lft}$ of G. The proof of their existence boils down to showing that the component group of $\mathcal{G}^{lft}_s$ is finitely generated (Proposition 3.5). We characterize the Néron model by a universal property in Definition 3.6.

Section 4 contains the basic results on jumps, the base change conductor, and the relation between them. In Section 4.1 we extend Edixhoven's results in [13] to arbitrary smooth commutative algebraic K-groups that admit a Néron model, and in particular to semi-abelian varieties. Section 4.2 briefly reviews Chai's elementary divisors and base change conductor for semi-abelian K-varieties, and in Section 4.3 we show that for tamely ramified semi-abelian K-varieties, the jumps and the elementary divisors are equivalent (Corollary 4.18).

In Section 5 we study the relation between jumps and monodromy eigenvalues for tamely ramified abelian K-varieties A. Theorem 5.5 computes the characteristic polynomial of the tame monodromy operator σ on the Tate module $T_\ell A$ in terms of the jumps of A. In Corollary 5.13, we prove that $c(A)$ corresponds to a monodromy eigenvalue on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$ as explained above.

Section 6 deals with the behaviour of $\mathcal{A}(d)^o_s$ when d varies over $\mathbb{N}'$. In Section 7 we express $ord_{\mathcal{A}(d)^o_s}(\omega(d))$ in terms of the jumps of A. In Section 8, we define the motivic zeta function of a semi-abelian K-variety, and we prove the global monodromy conjecture for tamely ramified abelian varieties (Theorem 8.5).

2. Preliminaries

2.1. **Notation.** We denote by R a discrete valuation ring, by K its quotient field, and by k its residue field. Additional conditions on R and k will be indicated at the beginning of each section. If R has equal characteristic, then we fix a k-algebra structure on R such that the composition $k \to R \to k$ is the identity. We denote by p the characteristic exponent of k, and we fix a uniformizer π in R. The choice of a value $|\pi|$ in $]0,1[$ determines a π-adic absolute value $|\cdot|$ on K. We denote by $\mathbb{N}'$ the set of strictly positive integers that are prime to p, and we fix a prime $\ell \neq p$. We fix a separable closure K^s of K, and we denote by K^t the tame closure of K in K^s.

We recall the following definition [6, 3.6.1]. A morphism of discrete valuation rings $R \to R'$ is called unramified if it is a flat local morphism, π is a uniformizer in R', and the extension of residue fields $R/(\pi) \to R'/(\pi)$ is separable. In this case, we call R' an unramified extension of R.

We denote by $K_0(Var_k)$ the Grothendieck ring of k-varieties, by $\mathbb{L}$ the class $[\mathbb{A}^1_k]$ of the affine line $\mathbb{A}^1_k$ in $K_0(Var_k)$, and by $\mathcal{M}_k$ the localization of $K_0(Var_k)$ w.r.t. $\mathbb{L}$. See for instance [24, 2.1]. We denote by

$$\chi_{top} : \mathcal{M}_k \to \mathbb{Z}$$

the unique ring morphism that sends the class $[X]$ of every k-variety X to the ℓ-adic Euler characteristic $\chi_{top}(X)$ of X. This morphism is independent of ℓ.

For every scheme S, we denote by (Sch/S) the category of S-schemes. If A is a commutative ring, then we write (Sch/A) instead of $(Sch/\operatorname{Spec} A)$. We denote by

$$(\cdot)_K : (Sch/R) \to (Sch/K) : X \mapsto X_K = X \times_R K$$

the generic fiber functor, and by

$$(\cdot)_s : (Sch/R) \to (Sch/k) : X \mapsto X_s = X \times_R k$$

the special fiber functor. For every R-scheme X and every section ψ in $X(R)$, we denote by $\psi(0)$ the image in X_s of the closed point of $\operatorname{Spec} R$.

For every scheme S and every group S-scheme G, we denote by $e_G \in G(S)$ the unit section, and we put $\omega_{G/S} = e_G^* \Omega^1_{G/S}$. If F is a field, then an algebraic F-group is a group F-scheme of finite type. We call a semi-abelian K-variety tamely ramified if it acquires semi-abelian reduction on a finite tame extension of K.

For every real number x we denote by $\lfloor x \rfloor$ the unique integer in $]x-1, x]$ and by $\lceil x \rceil$ the unique integer in $[x, x+1[$. We put $\{x\} = x - \lfloor x \rfloor \in [0,1[$. We denote by

$$\tau : \mathbb{Q} \to \mathbb{Z}_{>0}$$

the map that sends a rational number a to its order $\tau(a)$ in the quotient group $\mathbb{Q}/\mathbb{Z}$. For each $i \in \mathbb{Z}_{>0}$, we denote by $\Phi_i(t) \in \mathbb{Z}[t]$ the cyclotomic polynomial whose zeroes are the primitive i-th roots of unity.

If K is strictly henselian, then we adopt the following notations. We fix a topological generator σ of the tame monodromy group $G(K^t/K)$. For each $d \in \mathbb{N}'$ we denote by $K(d)$ the unique degree d extension of K in K^t, and by $R(d)$ the normalization of R in $K(d)$. For every algebraic K-variety X we put $X(d) = X \times_K K(d)$. For every differential form ω on X we denote by $\omega(d)$ the pull-back of ω to $X(d)$.

Finally, assume that R is complete. We call a formal R-scheme $\mathfrak{Y}$ $stft$ if it is separated and topologically of finite type over R. We denote by $\mathfrak{Y}_s$ its special

fiber (a separated k-scheme of finite type) and by $\mathfrak{Y}_\eta$ its generic fiber (a separated quasi-compact rigid K-variety). If k is separably closed, X is a rigid K-variety and ω a differential form on X, then $X(d)$ and $\omega(d)$ are defined as in the algebraic case. We denote by $(\cdot)^{an}$ the rigid analytic GAGA functor.

2.2. **Order of a gauge form.** Let X be a smooth R-scheme of pure relative dimension g, and let ω be a gauge form on X_K, i.e. a nowhere vanishing differential g-form on X_K. Let C be a connected component of X_s, and denote by η_C the generic point of C. Then $\mathcal{O}_{X,\eta_C}$ is a discrete valuation ring, whose maximal ideal is generated by π. Let ψ be a section in $X(R)$. Recall the following definitions.

Definition 2.1 (Order of a gauge form). *Choose an element α in $\mathbb{N}$ such that $\pi^\alpha \omega$ belongs to the image of the natural injection*

$$\Omega^g_{X/R}(X) \to \Omega^g_{X_K/K}(X_K)$$

We denote by ω' the unique inverse image of $\pi^\alpha \omega$ in $\Omega^g_{X/R}(X)$.

  The order of ω along C is defined by

$$ord_C(\omega) = \text{length}_{\mathcal{O}_{X,\eta_C}} \left((\Omega^g_{X/R})_{\eta_C}/\mathcal{O}_{X,\eta_C} \cdot \omega' \right) - \alpha$$

The order of ω at ψ is defined by

$$ord(\omega)(\psi) = \text{length}_R \psi^* \left(\Omega^g_{X/R}/\mathcal{O}_X \cdot \omega' \right) - \alpha$$

These definitions are independent of the choices of π and α.

  Note that $ord_C(\omega)$ and $ord(\omega)(\psi)$ are finite, since ω is a gauge form.

Proposition 2.2. *Let X be a smooth R-scheme of pure relative dimension, C a connected component of X_s, and ψ a section in $X(R)$ such that $\psi(0) \in C$. For every gauge form ω on X_K, we have*

$$ord_C(\omega) = ord(\omega)(\psi)$$

Proof. This property follows immediately from its formal counterpart [26, 5.10] and can easily be proven directly by similar arguments. $\square$

  Now assume that R is complete. If $\mathfrak{X}$ is a smooth $stft$ formal R-scheme, then the order of a gauge form ϕ on $\mathfrak{X}_\eta$ along a connected component C of the special fiber $\mathfrak{X}_s$ is defined as in the algebraic case [7, 4.3]. If $\mathfrak{X}$ is the formal π-adic completion of X, and ϕ is the restriction to $\mathfrak{X}_\eta$ of the rigid analytification ω^{an} on $(X_K)^{an}$, then we have

$$ord_C(\omega) = ord_C(\phi)$$

2.3. **Motivic integration on rigid varieties.** Assume that K is complete and that k is perfect. Let X be a separated smooth quasi-compact rigid K-variety of pure dimension, and let ω be a gauge form on X. The motivic integral

$$\int_X |\omega| \quad \in \mathcal{M}_k$$

was defined in [22, 4.1.2]. For our purposes, the following proposition can serve as a definition. The result is merely slightly stronger than [22, 4.3.1], but the difference is important for the applications in this paper. Recall that a weak Néron model for X is a smooth $stft$ formal R-scheme $\mathfrak{U}$ endowed with an open immersion of rigid K-varieties $i : \mathfrak{U}_\eta \to X$ such that $i(K') : \mathfrak{U}_\eta(K') \to X(K')$ is bijective for

every finite unramified extension K' of K [7, 1.3]. Such a weak Néron model always exists, by [7, 3.3].

Proposition 2.3. *For every weak Néron model* $(\mathfrak{U}, i)$ *of* X, *we have*

$$(2.1) \qquad \int_X |\omega| = \mathbb{L}^{-dim(X)} \sum_{C \in \pi_0(\mathfrak{U}_s)} [C] \mathbb{L}^{-ord_C(i^*\omega)} \quad \in \mathcal{M}_k$$

where $\pi_0(\mathfrak{U}_s)$ *denotes the set of connected components of* $\mathfrak{U}_s$. *In particular, the right hand side of (2.1) does not depend on* $\mathfrak{U}$.

Proof. This is almost the statement of [22, 4.3.1], except that there the additional condition was imposed that $\mathfrak{U}$ is an open formal subscheme of a formal model of X. Let us explain why it can be omitted. By [22, 4.3.1], the right hand side of (2.1) equals $\int_{\mathfrak{U}_\eta} |i^*\omega|$. Hence, it suffices to show that

$$\int_{\mathfrak{U}_\eta} |i^*\omega| = \int_X |\omega|$$

This follows from [31, 5.9]. □

If X does not have pure dimension, then by a gauge form ω on X, we mean the datum of a gauge form ω_Y on each connected component Y of X. The motivic integral of ω on X is then defined by

$$\int_X |\omega| = \sum_{Y \in \pi_0(X)} \int_Y |\omega_Y| \quad \in \mathcal{M}_k$$

2.4. **Motivic generating series.** Assume that R is complete and that k has characteristic zero. Let X be a separated smooth quasi-compact rigid K-variety, and ω a gauge form on X. We denote by $S(X, \omega; T)$ the motivic generating series

$$S(X, \omega; T) \in \mathcal{M}_k[[T]]$$

from [29, 7.2]. It depends on the choice of uniformizer π in general, but it is independent of this choice if k is algebraically closed [26, 4.10]. In fact, if k is algebraically closed, then

$$S(X, \omega; T) = \sum_{d>0} \left(\int_{X(d)} |\omega(d)| \right) T^d \quad \in \mathcal{M}_k[[T]]$$

In any case, the series $S(X, \omega; T)$ can be computed explicitly on a regular $stft$ formal R-model $\mathfrak{X}$ of X whose special fiber $\mathfrak{X}_s$ is a divisor with strict normal crossings [29, 7.7]. Such a model always exists, by embedded resolution of singularities for generically smooth $stft$ formal R-schemes [37, 3.4.1]. The explicit expression for the motivic generating series $S(X, \omega; T)$ shows in particular that $S(X, \omega; T)$ is rational and belongs to the subring

$$\mathcal{M}_k \left[\frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$$

of $\mathcal{M}_k[[T]]$.

2.5. **The monodromy conjecture.** In this section we assume that k has characteristic zero and that R is complete.

The local case. If $\mathfrak{X}$ is a regular *stft* formal R-scheme, then its motivic Weil generating series

$$S(\mathfrak{X}; T) \in \mathcal{M}_{\mathfrak{X}_s}[[T]]$$

was defined in [26, 7.33]. Here $\mathcal{M}_{\mathfrak{X}_s}$ is the localized Grothendieck ring of $\mathcal{M}_{\mathfrak{X}_s}$-varieties; see [24, 2.1]. The image of $S(\mathfrak{X}; T)$ under the forgetful morphism

$$\mathcal{M}_{\mathfrak{X}_s}[[T]] \to \mathcal{M}_k[[T]]$$

equals $S(\mathfrak{X}_\eta, \omega; T)$, with ω a so-called *Gelfand-Leray form* on $\mathfrak{X}_\eta$ [26, 7.21].

Conjecture 2.4 (Local Motivic Monodromy Conjecture)**.** *If $\mathfrak{X}$ is a regular stft formal R-scheme, then there exists a finite subset $\mathcal{S}$ of $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that*

$$S(\mathfrak{X}; T) \in \mathcal{M}_{\mathfrak{X}_s}\left[T, \frac{1}{1 - \mathbb{L}^a T^b}\right]_{(a,b) \in \mathcal{S}}$$

and such that for each $(a, b) \in \mathcal{S}$, the cyclotomic polynomial $\Phi_{\tau(a/b)}(t)$ divides the characteristic polynomial of σ on $R^i \psi_{\mathfrak{X}}(\mathbb{Q}_\ell)_x$, for some $i \in \mathbb{Z}_{\geq 0}$ and some geometric closed point x of $\mathfrak{X}_s$.

Here $\psi_{\mathfrak{X}}$ denotes the nearby cycle functor on the formal R-scheme $\mathfrak{X}$ (called vanishing cycle functor in [4][5]). In particular, Conjecture 2.4 implies that for each pole α of $S(\mathfrak{X}; \mathbb{L}^{-s})$ and for every embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, the value $\exp(2\pi i \alpha)$ is an eigenvalue of the monodromy action of σ on $R^i \psi_{\mathfrak{X}}(\mathbb{Q}_\ell)_x$, for some $i \in \mathbb{Z}_{\geq 0}$ and some geometric closed point x of $\mathfrak{X}_s$. By the comparison result in [26, 9.6], Conjecture 2.4 implies the following conjecture of Denef and Loeser's, which at its turn generalizes Igusa's monodromy conjecture for p-adic zeta functions [11, § 2.4].

Conjecture 2.5 (Denef and Loeser's Motivic Monodromy Conjecture)**.** *Let k be a subfield of $\mathbb{C}$, and let X be a smooth and irreducible k-variety endowed with a dominant morphism $f : X \to \mathbb{A}^1_k$. Denote by X_s the special fiber of f and by*

$$Z_f(T) \in \mathcal{M}_{X_s}[[T]]$$

the motivic zeta function associated to f [12, 3.2.1]. Then there exists a finite subset $\mathcal{S}$ of $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that

$$Z_f(T) \in \mathcal{M}_{X_s}\left[T, \frac{1}{1 - \mathbb{L}^a T^b}\right]_{(a,b) \in \mathcal{S}}$$

and such that for each $(a, b) \in \mathcal{S}$, the value $\exp(2\pi i a/b)$ is an eigenvalue of monodromy on $R^i \psi_f(\mathbb{Q})_x$ for some $i \in \mathbb{Z}_{\geq 0}$ and some $x \in X_s(\mathbb{C})$. Here $R\psi_f(\mathbb{Q}) \in D^b_c(X_s(\mathbb{C}), \mathbb{Q})$ denotes the complex-analytic nearby cycles complex of f.

We refer to [10][27][30] for an introduction to the p-adic and motivic monodromy conjecture.

Remark. To be precise, Denef and Loeser's conjecture is a bit stronger than Conjecture 2.5, because it is stated for the "monodromic" motivic zeta function which carries an additional action of the pro-k-group $\widehat{\mu}$ of roots of unity [12, 3.2.1]. The zeta function $Z_f(T)$ in Conjecture 2.5 is the image of this monodromic zeta function under the forgetful morphism

$$\mathcal{M}^{\widehat{\mu}}_{X_s}[[T]] \to \mathcal{M}_{X_s}[[T]]$$

$\square$

The global case. The following definition formulates a global version of the motivic monodromy conjecture.

Definition 2.6. *Let X be a smooth, proper, geometrically connected K-variety, and assume that X admits a gauge form ω. We say that X satisifes the Global Monodromy Property (GMP) if there exists a finite subset $\mathcal{S}$ of $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that*

$$S(X, \omega, T) \in \mathcal{M}_k \left[T, \frac{1}{1 - \mathbb{L}^a T^b} \right]_{(a,b) \in \mathcal{S}}$$

and such that for each $(a, b) \in \mathcal{S}$, the cyclotomic polynomial $\Phi_{\tau(a/b)}(t)$ divides the characteristic polynomial of σ on $H^i(X \times_K K^s, \mathbb{Q}_\ell)$ for some $i \in \mathbb{N}$.

Note that the property (GMP) only depends on X, and not on ω, since we have

$$S(X, u \cdot \omega; T) = S(X, \omega; \mathbb{L}^{-v_K(u)} T) \in \mathcal{M}_k[[T]]$$

for all $u \in K^*$, where v_K denotes the discrete valuation on K^*.

Question 2.7. *Is there a natural condition on X that guarantees that X satisfies the Global Monodromy Property (GMP)?*

We will show in Theorem 8.5 that if k is algebraically closed, every abelian K-variety satisfies the Global Monodromy Property. Moreover, we will show that this result extends to tamely ramified abelian varieties in mixed and positive characteristic. Our proof does not use resolution of singularities.

3. Néron models

Let G be a smooth commutative algebraic K-group. The notion of Néron lft-model for G over R is defined in [6, 10.1.1]. It is shown in [6, 10.2.2] that G admits a Néron lft-model iff $G \times_K \widehat{K}$ does not contain a subgroup of type $\mathbb{G}_{a, \widehat{K}}$, where we denote by $\widehat{K}$ the completion of K. In particular, every semi-abelian K-variety A admits a Néron lft-model over R.

Lemma 3.1. *Assume that k is separably closed. Let G be a smooth commutative algebraic K-group. Then G admits a Néron lft-model $\mathcal{G}^{lft}$ iff G^o admits a Néron lft-model $\mathcal{H}^{lft}$. If we denote by f the unique morphism $\mathcal{H}^{lft} \to \mathcal{G}^{lft}$ of group R-schemes extending the open immersion $G^o \to G$, then the morphism of constant groups $\pi_0(f_s) : \pi_0(\mathcal{H}_s^{lft}) \to \pi_0(\mathcal{G}_s^{lft})$ is an injection whose cokernel is canonically isomorphic to $G(K)/G^o(K)$.*

Proof. It is clear from the definition that G admits a Néron lft-model iff G^o does, and that the formation of Néron lft-models commutes with finite disjoint unions, so that f is an open and closed immersion. If C is a connected component of G without K-point, then C is a Néron lft-model of itself, with empty special fiber. The connected components of G with K-point form a constant subgroup scheme of $\pi_0(G)$, which is canonically isomorphic to $G(K)/G^o(K)$. For any connected component C of $\mathcal{G}_s^{lft}$, there exists a unique connected component C' of G such that C belongs to the schematic closure of C' in $\mathcal{G}^{lft}$, and $C'(K) \neq \emptyset$. The map $C \mapsto C'$ defines a surjection $\pi_0(\mathcal{G}_s^{lft}) \to G(K)/G^o(K)$ whose kernel is precisely $\pi_0(\mathcal{H}_s^{lft})$. $\square$

Lemma 3.2. *Assume that K is Henselian. Let G be a smooth commutative algebraic K-group, and let K' be a finite separable extension of K. Then G admits a Néron lft-model iff $G \times_K K'$ admits a Néron lft-model.*

Proof. By Lemma 3.1 we may assume that G is connected. By [6, 10.2.2] it suffices to show that $G \times_K \widehat{K}$ admits a subgroup of type $\mathbb{G}_{a,\widehat{K}}$ iff $G \times_K \widehat{K'}$ admits a subgroup of type $\mathbb{G}_{a,\widehat{K'}}$. The "only if" part is obvious, so let us prove the "if" part. Since $\widehat{K'}$ is a finite separable extension of $\widehat{K}$, we may assume that K is complete. Assume that G does not admit a subgroup of type $\mathbb{G}_{a,K}$. Let T be the maximal torus in G, and put $H = G/T$. It is shown in the proof of [6, 10.2.2] that $H \times_K K'$ does not admit a subgroup of type $\mathbb{G}_{a,K'}$. Assume that $G \times_K K'$ has a subgroup isomorphic to $\mathbb{G}_{a,K'}$. Then the restriction of $f : G \times_K K' \to H \times_K K'$ to $\mathbb{G}_{a,K'}$ must have a non-trivial kernel U, which is contained in $ker(f) = T \times_K K'$. But U is unipotent, and there are no non-trivial morphisms of group K'-schemes $U \to T \times_K K'$ [2, XVII.2.4], so we arrive at a contradiction. $\square$

Lemma 3.3. *Assume that K is Henselian. Let K' be a finite separable extension of K, and denote by R' the normalization of R in K'. Let G be a smooth commutative algebraic K-group that admits a Néron lft model $\mathcal{G}^{lft}$, and denote by $(\mathcal{G}')^{lft}$ the Néron lft-model of $G' = G \times_K K'$. Consider the unique morphism of group R'-schemes*

$$f : \mathcal{G}^{lft} \times_R R' \to (\mathcal{G}')^{lft}$$

that extends the canonical isomorphism between the generic fibers. This morphism f is of finite type.

Proof. Consider the morphism of group R-schemes

$$g : \mathcal{G}^{lft} \to W := \prod_{R'/R} (\mathcal{G}')^{lft}$$

obtained from f by adjunction, and denote by $\mathcal{H}$ the schematic closure of G_K in W_K. It is a closed subgroup scheme of W. Since $\mathcal{G}^{lft}$ is flat, the morphism g factors through a morphism of group R-schemes $g' : \mathcal{G}^{lft} \to \mathcal{H}$. By [6, 10.1.4], this is the canonical group smoothening of $\mathcal{H}$. In particular, it is of finite type. This implies that f is of finite type, since it is the composition of $g \times_R R'$ with the tautological morphism of finite type $W \times_R R' \to (\mathcal{G}')^{lft}$. $\square$

Lemma 3.4. *Let G be a connected smooth commutative algebraic K-group that admits a Néron lft model $\mathcal{G}^{lft}$. Assume that the maximal torus T in G is split. Then the component group $\pi_0(\mathcal{G}_s^{lft} \times_k k^s)$ is finitely generated.*

Proof. Since the formation of Néron lft-models commutes with unramified base change [6, 10.1.3] we may assume that K is complete and strictly henselian. Denote the Néron lft-model of T by $\mathcal{T}^{lft}$, and denote by H the quotient G/T. Then H has a Néron lft-model $\mathcal{H}^{lft}$ which is of finite type over R, by the proof of [6, 10.2.2]. By the proof of [6, 10.1.7], the sequence

$$\pi_0(\mathcal{T}_s^{lft}) \to \pi_0(\mathcal{G}_s^{lft}) \to \pi_0(\mathcal{H}_s^{lft}) \to 1$$

is exact. Since $\pi_0(\mathcal{T}_s^{lft})$ is a free $\mathbb{Z}$-module whose rank is equal to the dimension of T and $\pi_0(\mathcal{H}_s^{lft})$ is finite, we see that $\pi_0(\mathcal{G}_s^{lft})$ is finitely generated. $\square$

Proposition 3.5. *Let G be a smooth commutative algebraic K-group, and assume that G admits a Néron lft-model $\mathcal{G}^{lft}$. Then the component group $\pi_0(\mathcal{G}_s^{lft} \times_k k^s)$ is finitely generated. In particular, its torsion part is finite.*

Proof. Since the formation of Néron lft-models commutes with unramified base change [6, 10.1.3] we may assume that K is strictly Henselian. We may also assume that G is connected, by Lemma 3.1. Let T be the maximal torus in G. Let K' be a finite separable extension of K such that T splits over K'. Denote by R' the normalization of R in K'. By Lemma 3.2, $G \times_K K'$ admits a Néron lft-model $(\mathcal{G}')^{lft}$. By Lemma 3.4, the component group $\pi_0((\mathcal{G}')_s^{lft})$ is finitely generated. By Lemma 3.3, the natural morphism of group R'-schemes

$$f : \mathcal{G}^{lft} \times_R R' \to (\mathcal{G}')^{lft}$$

is of finite type, so that the induced morphism of component groups

$$\pi_0(\mathcal{G}_s^{lft}) \to \pi_0((\mathcal{G}')_s^{lft})$$

has finite kernel. Hence, $\pi_0(\mathcal{G}_s^{lft})$ is finitely generated. $\qquad\square$

Definition 3.6. *For every commutative ring S, we denote by $(\mathscr{G}r/S)$ the category of smooth separated group S-schemes of finite type. We denote by $(Groups)$ the category of groups.*

Let G be a smooth algebraic K-group. If the functor

$$NM_G : (\mathscr{G}r/R) \to (Groups) : H \mapsto Hom_{(\mathscr{G}r/K)}(H_K, G)$$

is representable by an object $\mathcal{G}$ of $(\mathscr{G}r/R)$, then we call $\mathcal{G}$ the Néron model of G. We denote by Φ_G the group k-scheme of connected components $\pi_0(\mathcal{G}_s)$, and we denote its rank by $\phi(G)$.

It follows immediately from the definition that there exists a canonical isomorphism of algebraic K-groups $G \cong \mathcal{G}_K$. If G admits a Néron model in the sense of [6] (i.e., if its Néron lft-model exists and is quasi-compact) then this Néron model represents the functor NM_G, so that our definition includes the one in [6].

Proposition 3.7. *Let G be a smooth commutative algebraic K-group, and assume that G has a Néron lft-model $\mathcal{G}^{lft}$. Then G has a Néron model $\mathcal{G}$. The canonical isomorphism of group K-schemes $\mathcal{G}_K \cong G$ extends uniquely to a morphism of group R-schemes*

$$\varphi : \mathcal{G} \to \mathcal{G}^{lft}$$

The morphism φ is an open immersion, and its image is the maximal quasi-compact open subgroup scheme of $\mathcal{G}^{lft}$. The group k-scheme Φ_G is the torsion part of $\pi_0(\mathcal{G}_s^{lft})$.

Proof. Denote by $\mathcal{G}$ the union of the generic fiber G of $\mathcal{G}^{lft}$ with all connected components C of the special fiber $\mathcal{G}_s^{lft}$ such that C defines a torsion point of $\pi_0(\mathcal{G}_s^{lft})$. Then $\mathcal{G}$ is an open subgroup scheme of $\mathcal{G}^{lft}$. Moreover, since $\pi_0(\mathcal{G}_s^{lft} \times_k k^s)$ has finite torsion part by Proposition 3.5, $\mathcal{G}$ is quasi-compact and hence of finite type over R. It is clear that $\mathcal{G}$ is the largest quasi-compact open subgroup scheme of $\mathcal{G}^{lft}$.

Now we show that $\mathcal{G}$ represents the functor NM_G. Let H be a smooth group scheme of finite type over R, and consider a morphism of group K-schemes $H_K \to G$. By the universal property of the Néron lft-model, this morphism extends uniquely to a morphism of group R-schemes $h : H \to \mathcal{G}^{lft}$. Since H is

quasi-compact, the image of $\pi_0(H_s \times_k k^s)$ in $\pi_0(\mathcal{G}_s^{lft} \times_k k^s)$ is a finite subgroup of $\pi_0(\mathcal{G}_s^{lft} \times_k k^s)$. Hence, the image of h is contained in $\mathcal{G}$. $\qquad\square$

Corollary 3.8. *If A is a semi-abelian K-variety, then A admits a Néron model.*

Definition 3.9. *Assume that K is complete. Let G be a smooth commutative algebraic K-group that admits a Néron model $\mathcal{G}$. We define the bounded part G^b of G as the generic fiber of the formal π-adic completion of $\mathcal{G}$. It is a separated, smooth, quasi-compact open rigid subgroup of the rigid analytification G^{an} of G.*

Proposition 3.10. *Assume that R is excellent. Let G be a smooth commutative algebraic K-group. Then G admits a Néron model iff G admits a Néron lft-model.*

Proof. If G admits a Néron lft-model, then G admits a Néron model by Proposition 3.7. Conversely, assume that G admits a Néron model $\mathcal{G}$. By [6, 10.2.2] it is enough to show that G does not contain a subgroup of type $\mathbb{G}_{a,K} = \operatorname{Spec} K[\xi]$.

Suppose that G admits a subgroup of type $\mathbb{G}_{a,K}$. Since any subgroup scheme of G is closed [1, VI_B 1.4.2], the set

$$B = \mathcal{G}(R) \cap \mathbb{G}_{a,K}(K) \subset \mathbb{G}_{a,K}(K)$$

is bounded in $\mathbb{G}_{a,K}$ in the sense of [6, 1.1.2]. Therefore, B is contained in a closed disc in $\mathbb{G}_{a,K}(K)$ defined by $|\xi| \leq |\pi|^{-N}$ for some $N \in \mathbb{N}$.

Now consider the smooth group R-scheme of finite type

$$H = \mathbb{G}_{a,R} = \operatorname{Spec} R[\zeta]$$

and the morphism of group K-schemes

$$h : H_K \to \mathbb{G}_{a,K} : \xi \mapsto \pi^{-(N+1)}\zeta$$

By the universal property of the Néron model, the induced morphism $H_K \to G$ extends to an R-morphism $H \to \mathcal{G}$. This is a contradiction, since the point x of $H_K(K)$ defined by the ideal $(\zeta - 1)$ belongs to $H(R)$, while it is mapped to the point $(\xi - \pi^{-(N+1)})$ in $\mathbb{G}_{a,K}(K)$, which does not belong to B. $\qquad\square$

Proposition 3.11. *Let $R \to S$ be an unramified morphism of discrete valuation rings and denote by L the quotient field of S. Let G be a smooth commutative algebraic K-group. If G admits a Néron lft-model $\mathcal{G}^{lft}$, and $\mathcal{G}$ is a Néron model of G, then $\mathcal{G} \times_R S$ is a Néron model for $G \times_K L$.*

Proof. By [6, 10.1.3], $\mathcal{G}^{lft} \times_R S$ is a Néron lft-model for $G \times_K L$. Since the torsion part of $\pi_0(\mathcal{G}_s^{lft})$ is stable under the base change $k \to S/(\pi)$, it follows from Proposition 3.7 that $\mathcal{G} \times_R S$ is a Néron model of $G \times_K L$. $\qquad\square$

Proposition 3.12. *Assume that R is excellent. Let G be a smooth commutative algebraic K-group, and assume that G has a Néron model $\mathcal{G}$. Let $R \to S$ be an unramified morphism of discrete valuation rings and denote by L the quotient field of S. Then $\mathcal{G}(S) \subset G(L)$ is the unique maximal subgroup of $G(L)$ that is bounded in G in the sense of [6, 1.1.2].*

Proof. By Propositions 3.10 and 3.11, and [6, 1.1.5], we may assume that $R = S$. The group $\mathcal{G}(R)$ is bounded in G by [6, 1.1.7]. Conversely, let x be a point of $G(K)$ and assume that x belongs to a subgroup of $G(K)$ that is bounded in G. Then the subgroup $<x>$ of $G(K)$ generated by x is bounded in G. We will show that $x \in \mathcal{G}(R)$. By Proposition 3.10, G has a Néron lft-model $\mathcal{G}^{lft}$. The point x extends

uniquely to a section ψ in $\mathcal{G}^{lft}(R)$, and it suffices to show that ψ_s is contained in $\mathcal{G}_s(k) \subset \mathcal{G}_s^{lft}(k)$.

Since $<x>$ is bounded in G, there exists a smooth quasi-compact R-model X of G such that the image of the natural map $X(R) \to G(K)$ contains $<x>$, by [6, 1.1.8+3.5.2]. By the universal property of the Néron lft-model, there exists a unique R-morphism $h : X \to \mathcal{G}^{lft}$ extending the isomorphism $X_K \cong G$. Since X is quasi-compact, the morphism h_s factors through a finite union of connected components of $\mathcal{G}_s^{lft}$. On the other hand, the image of $h_s(k) : X_s(k) \to \mathcal{G}_s^{lft}(k)$ contains the subgroup of $\mathcal{G}_s^{lft}(k)$ generated by ψ_s, so the connected component of $\mathcal{G}_s^{lft}$ containing ψ_s is torsion in $\pi_0(\mathcal{G}_s^{lft})$, and we have $\psi_s \in \mathcal{G}_s(k)$ by Proposition 3.7. $\qquad\square$

In particular, if K is complete, then $G^b(K')$ is the maximal bounded subgroup of $G(K')$, for every finite unramified extension K' of K.

Remark. We do not know if the condition that R is excellent is necessary in Propositions 3.10 and 3.12. If Proposition 3.11 holds without the condition that G admits a Néron lft-model, i.e., if Néron models always commute with unramified base change, then the excellence condition can be omitted in Propositions 3.10 and 3.12, since it follows from [6, 10.2.2] that G admits a Néron lft-model iff $G \times_K \widehat{K}$ admits one. $\qquad\square$

4. Edixhoven's filtration and Chai's base change conductor

In this section, we assume that K is strictly Henselian. In [13], Edixhoven constructed a filtration on the special fiber of the Néron model of an abelian K-variety, which measures the behaviour of the Néron model under finite tame extensions of K. This construction generalizes without additional effort to the class of smooth and commutative algebraic K-groups G such that $G \times_K K'$ admits a Néron model for all finite tame extensions K' of K, and in particular to the class of semi-abelian K-varieties. The construction is explained in Section 4.1, rephrased in the language of Greenberg schemes. Most of the results in Section 4.1 were stated (for abelian varieties) in [13], but many proofs were omitted. Since these results are vital for the applications in this article, and since some of them don't seem trivial to us (in particular Theorem 4.10), we found it worthwhile to supply detailed proofs here if they were not given in [13]. If G is a tamely ramified semi-abelian K-variety, then we relate Edixhoven's filtration to Chai's base change conductor [8] in Sections 4.2 and 4.3.

4.1. **Edixhoven's filtration.** Let K' be a tame finite extension of K of degree d, and denote by R' the normalization of R in K'. For each integer $n > 0$, we put $R'_n = R'/(\mathfrak{M}')^n$, where $\mathfrak{M}'$ is the maximal ideal of R'. We put $R'_0 = \{0\}$, the trivial R'-module. If $1 \le n \le d$, then R'_n carries a natural k-algebra structure $k \cong R/\mathfrak{M} \to R'/(\mathfrak{M}')^n$.

Let G be a smooth commutative algebraic K-group such that $G' = G \times_K K'$ admits a Néron model $\mathcal{G}'$. We denote by X the Weil restriction $\prod_{R'/R} \mathcal{G}'$ of $\mathcal{G}'$ to R. By [6, 7.6.4] it is representable by a group R-scheme, since R' is finite and flat over R and $\mathcal{G}'$ is quasi-projective over R' [6, 6.4.1]. The R-scheme X is separated, smooth, and of finite type over R [6, 7.6.5].

The extension K'/K is Galois. We denote its Galois group by μ, and we let μ act on K' from the left. The action of $\zeta \in \mu$ on $\mathfrak{M}'/(\mathfrak{M}')^2$ is multiplication by ζ', for some $\zeta' \in k$, and the map $\zeta \mapsto \zeta'$ is an isomorphism between μ and the group $\mu_d(k)$ of d-th roots of unity in k. By the universal property of the Néron model, the right μ-action on G' extends uniquely to a μ-action on the scheme $\mathcal{G}'$ such that the structural morphism $\mathcal{G}' \to \operatorname{Spec} R'$ is μ-equivariant. As in [13, 2.4], this action induces a right μ-action on X.

The fixed point functor $(\cdot)^\mu$ from the category of schemes with right μ-action to the category of schemes is right adjoint to the functor endowing a scheme with the trivial μ-action. We refer to [13, 3.1] for its basic properties.

We have a tautological morphism of K-groups

$$G \to X_K \cong \prod_{K'/K} G'$$

By Galois descent, it factors through an isomorphism

$$G \to (X_K)^\mu \cong (X^\mu)_K$$

Proposition 4.1. *The R-scheme X^μ is a Néron model for G.*

Proof. By [13, 3.4] the group R-scheme X^μ is smooth. Let H be any smooth group R-scheme of finite type, and let

$$f : H_K \to (X^\mu)_K$$

be a morphism of group K-schemes. Then f corresponds to a μ-equivariant K'-morphism $f : H \times_R K' \to G'$ where μ acts on $H \times_R K'$ via the Galois action on K'. By the universal property of the Néron model, f extends to a μ-equivariant morphism of group R'-schemes $f' : H \times_R R' \to \mathcal{G}'$, which yields a morphism $g : H \to X$. If we let μ act trivially on H, then g is μ-equivariant, so that g factors through a morphism of group R-schemes $g : H \to X^\mu$ extending f. $\square$

Definition 4.2. *For any R'-scheme Y and any $i \in \{1, \dots, d\}$ we put*

$$Green_i(Y) = \prod_{R'_i/k} (Y \times_{R'} R'_i)$$

For $j \geq i$ in $\{1, \dots, d\}$ the truncation morphism $R'_j \to R'_i$ induces a morphism of k-schemes

$$\theta^j_i : Green_j(Y) \to Green_i(Y)$$

Since $\operatorname{Spec} R'_i \to \operatorname{Spec} k$ is universally bijective, the proof of [6, 7.6.4] shows that $Green_i(Y)$ is indeed representable by a k-scheme. The functor $Green_i(\cdot)$ is compatible with open immersions. Note that the k-scheme $Green_i(\mathcal{G}')$ inherits a group structure from $\mathcal{G}'$, as well as a right μ-action. The truncation morphisms θ^j_i are μ-equivariant morphisms of group k-schemes.

Remark. If k is perfect or R has equal characteristic, then by [31, 4.1], $Green_i(\mathcal{G}')$ is canonically isomorphic to the Greenberg scheme $Gr^{R'}_{i-1}(\mathcal{G}')$ of $\mathcal{G}'$ (mind the shift of index in our notation R'_i w.r.t. [31]). $\square$

Definition 4.3. *For $i \in \{1, \dots, d\}$ we define $F^i X_s$ as the kernel of*

$$\theta^d_i : X_s \cong Green_d(\mathcal{G}') \to Green_i(\mathcal{G}')$$

This defines a decreasing filtration

$$X_s =: F^0 X_s \supset F^1 X_s \supset \ldots \supset F^d X_s = 0$$

by subgroup k-schemes that are stable under the μ-action.

For $i \in \{0, \ldots, d-1\}$, *we denote by* $Gr^i X_s$ *the group k-scheme* $F^i X_s / F^{i+1} X_s$. *It inherits a μ-action from* $F^i X_s$.

Proposition 4.4. *Denote by m the dimension of G. There exists a Zariski cover $\mathcal{U}$ of $\mathcal{G}'$ such that for each member U of $\mathcal{U}$ and each pair of integers $j \geq i$ in $\{1, \ldots, d\}$, the truncation morphism*

$$\theta_i^j : Green_j(U) \to Green_i(U)$$

is a trivial fibration whose fiber is isomorphic (as a k-scheme) to $\mathbb{A}_k^{(j-i)m}$.

For each $i \in \{1, \ldots, d-1\}$, *the kernel of θ_i^{i+1} is canonically isomorphic to $Gr^i X_s$. Moreover, the group k-scheme $Gr^0 X_s$ is canonically isomorphic to $\mathcal{G}'_s$.*

Proof. The fact that θ_i^j is a trivial fibration with fiber $\mathbb{A}_k^{(j-i)m}$ above the members of a Zariski cover of $\mathcal{G}'$ can be proven exactly as in the case where k is perfect [35, 3.4.2]. For the second part of the statement, consider the short exact sequence of group k-schemes

$$0 \longrightarrow F^i X_s \longrightarrow X_s = Green_d(\mathcal{G}') \xrightarrow{\theta_i^d} Green_i(\mathcal{G}') \longrightarrow 0$$

Dividing by $F^{i+1} X_s$ we obtain an exact sequence

$$0 \longrightarrow Gr^i X_s \longrightarrow X_s / F^{i+1} X_s \xrightarrow{\overline{\theta}_i^d} Green_i(\mathcal{G}') \longrightarrow 0$$

However, by the first exact sequence (with i replaced by $i+1$) we see that there exists a canonical isomorphism $X_s / F^{i+1} X_s \cong Green_{i+1}(\mathcal{G}')$ identifying $\overline{\theta}_i^d$ with θ_i^{i+1}. Considering the short exact sequence

$$0 \longrightarrow F^1 X_s \longrightarrow X_s \xrightarrow{\theta_1^d} Green_1(\mathcal{G}') \cong \mathcal{G}'_s \longrightarrow 0$$

we see that $Gr^0 X_s \cong \mathcal{G}'_s$. $\square$

Proposition 4.5. *For each $0 < i < d$, there exists a canonical μ-equivariant isomorphism of group k-schemes*

$$Gr^i X_s \cong Lie(\mathcal{G}'_s) \otimes_k (\mathfrak{M}'/(\mathfrak{M}')^2)^{\otimes i}$$

where we view the right hand side as a vector group k-scheme, and where the right action of μ on $(\mathfrak{M}'/(\mathfrak{M}')^2)^{\otimes i}$ is the inverse of the left Galois action. In particular, $F^i X_s$ is unipotent, smooth and connected for $0 < i < d$.

Proof. The proof of [13, §5.1] carries over without changes. If k is perfect or R has equal characteristic, see also [14, §2]. $\square$

Definition 4.6. *We define a decreasing filtration on $\mathcal{G}_s$ by subgroup k-schemes*

$$\mathcal{G}_s = F_d^0 \mathcal{G}_s \supset F_d^1 \mathcal{G}_s \supset \ldots \supset F_d^d \mathcal{G}_s = 0$$

by putting $F_d^i \mathcal{G}_s = (F^i X_s)^\mu$. For each $i \in \{0, \ldots, d-1\}$, we put

$$Gr_d^i \mathcal{G}_s = F_d^i \mathcal{G}_s / F_d^{i+1} \mathcal{G}_s$$

Note that we indeed have $\mathcal{G}_s = (F^0 X_s)^\mu = (X_s)^\mu$ by Proposition 4.1. It follows immediately from the definition that $F_d^i \mathcal{G}_s$ is the kernel of the truncation morphism

$$\mathcal{G}_s \cong (X_s)^\mu \to Green_i(\mathcal{G}')$$

for $i = 1, \ldots, d$. Observe that, since the extension K' of K of degree $d \in \mathbb{N}'$ is uniquely determined by d up to K-automorphism, the filtration $F_d^\bullet \mathcal{G}_s$ only depends on d and not on K'.

Lemma 4.7. *For each $i \in \{0, \ldots, d-1\}$ there is a canonical isomorphism*

$$Gr_d^i \mathcal{G}_s \cong (Gr^i X_s)^\mu$$

Proof. It suffices to show that the exact sequence

$$0 \to F^{i+1} X_s \to F^i X_s \to Gr^i X_s \to 0$$

remains exact after applying the fixed point functor $(\cdot)^\mu$, for each $i \in \{0, \ldots, d-1\}$.

Left exactness is clear. It remains to show that $(F^i X_s)^\mu \to (Gr^i X_s)^\mu$ is a surjection of $fpqc$ sheaves. For any commutative group k-scheme H, we denote by $d_H : H \to H$ the multiplication by d. By Proposition 4.5, $F^{i+1} X_s$ is unipotent, so since d is invertible in k, multiplication by d is an automorphism on $F^{i+1} X_s$. We denote its inverse by $(d^{-1})_{F^{i+1} X_s}$.

For any commutative group k-scheme Z endowed with a right μ-action, consider the morphism $N_Z : Z \to Z^\mu$ defined by

$$N_Z(S) : Z(S) \to Z^\mu(S) : s \mapsto \sum_{\zeta \in \mu} s * \zeta$$

for all k-schemes S. If we denote by ι_Z the tautological closed immersion $Z^\mu \to Z$, then $N_Z \circ \iota_Z = d_{Z^\mu}$.

Let S be a k-scheme, and c a section in $(Gr^i X_s)^\mu(S)$. Choose an $fpqc$ covering $S' \to S$ such that c lifts to an element b of $(F^i X_s)(S')$. Since c is invariant under the μ-action, the element

$$b' = N_{F^i X_s}(b) - d_{F^i X_s}(b)$$

maps to zero in $(Gr^i X_s)(S')$, so it belongs to $(F^{i+1} X_s)(S')$. Now

$$b'' = b + (d^{-1})_{F^{i+1} X_s}(b')$$

maps to c. This element belongs to $(F^i X_s)^\mu(S')$, since for any $\zeta \in \mu$, $b'' * \zeta - b''$ is an element of $(F^{i+1} X_s)(S')$ that is mapped to zero by $d_{F^{i+1} X_s}$, so that $b'' * \zeta = b''$. $\square$

Corollary 4.8. *The group k-scheme $Gr_d^0 \mathcal{G}_s$ is canonically isomorphic to $(\mathcal{G}'_s)^\mu$. It is a smooth algebraic k-group, and there is a canonical isomorphism of k-vector spaces*

$$Lie((\mathcal{G}'_s)^\mu) \cong (Lie(\mathcal{G}'_s))^\mu$$

For each $0 < i < d$, there exists a canonical isomorphism

(4.1) $$Gr_d^i \mathcal{G}_s \cong Lie(\mathcal{G}'_s)[i] \otimes_k (\mathfrak{M}'/(\mathfrak{M}')^2)^{\otimes i}$$

where $Lie(\mathcal{G}'_s)[i]$ is the subspace of $Lie(\mathcal{G}'_s)$ where $\mu \cong \mu_d(k)$ acts as $(v, \zeta) \mapsto \zeta^i \cdot v$ for $(v, \zeta) \in Lie(\mathcal{G}'_s) \times \mu_d(k)$, and where we view the right hand side of (4.1) as a vector group k-scheme. In particular, $F_d^i \mathcal{G}_s$ is unipotent, smooth and connected for $0 < i \leq d$.

Proof. Smoothness of $(\mathcal{G}'_s)^\mu$ and the isomorphism

$$Lie((\mathcal{G}'_s)^\mu) \cong (Lie(\mathcal{G}'_s))^\mu$$

follow from [13, 3.2+4]. The remaining statements follow immediately from Proposition 4.5 and Lemma 4.7. □

Definition 4.9. *We say that an integer j in $\{0, \ldots, d-1\}$ is a K'-jump of G if $Gr_d^j \mathcal{G}_s \neq 0$. The dimension of $Gr_d^j \mathcal{G}_s$ is called the multiplicity of the jump j.*

By Corollary 4.8, the K'-jumps of G and their multiplicites can be computed from the μ-action on $Lie(\mathcal{G}'_s)$.

The following theorem will play a crucial role in the remainder of this article. If G is an abelian variety, the result was stated in [13, 5.4.6] without proof.

Theorem 4.10. *Let K' be a finite tame extension of K, and denote by R' the normalization of R in K'. Let G be a smooth commutative algebraic K-group, and assume that $G' = G \times_K K'$ admits a Néron model $\mathcal{G}'$. Denote by $\mathcal{G}$ the Néron model of G, and by $\mathcal{K}$ the kernel of the natural morphism $h : \mathcal{G} \times_R R' \to \mathcal{G}'$. Let $j_1(G, K'), \ldots, j_u(G, K')$ be the K'-jumps associated to G, with respective multiplicities $m_1(G, K'), \ldots, m_u(G, K')$.*

Then the pull-back of the fundamental exact sequence of $\mathcal{O}_{\mathcal{G} \times_R R'}$-modules

$$h^* \Omega^1_{\mathcal{G}'/R'} \to \Omega^1_{\mathcal{G} \times_R R'/R'} \to \Omega^1_{\mathcal{G} \times_R R'/\mathcal{G}'} \to 0$$

w.r.t. the unit section $e_{\mathcal{G} \times_R R'}$ yields a short exact sequence of R'-modules

(4.2) $$0 \to \omega_{\mathcal{G}'/R'} \to \omega_{\mathcal{G} \times_R R'/R'} \to \omega_{\mathcal{K}/R'} \to 0$$

and $\omega_{\mathcal{K}/R'}$ is isomorphic to

$$\left(\bigoplus_{m_1(G,K')} R'_{j_1(G,K')} \right) \oplus \cdots \oplus \left(\bigoplus_{m_u(G,K')} R'_{j_u(G,K')} \right)$$

Moreover, $Lie(h)$ injects $Lie(\mathcal{G} \times_R R')$ into $Lie(\mathcal{G}')$ and there exists an isomorphism of R'-modules

(4.3) $$\omega_{\mathcal{K}/R'} \cong Lie(\mathcal{G}')/Lie(\mathcal{G} \times_R R')$$

Proof. We refer to [20, §1] for some basic results on Lie algebras of group schemes. To show that the sequence (4.2) is exact, it suffices to show that

$$\omega_{\mathcal{G}'/R'} \to \omega_{\mathcal{G} \times_R R'/R'}$$

is injective. This follows immediately from the fact that $\omega_{\mathcal{G}'/R'}$ is free and $h_{K'}$ an isomorphism. Dualizing (4.2) we find an exact sequence

$$0 \longrightarrow Lie(\mathcal{G} \times_R R') \xrightarrow{Lie(h)} Lie(\mathcal{G}') \longrightarrow Ext^1_{R'}(\omega_{\mathcal{K}/R'}, R') \longrightarrow 0$$

Since $\omega_{\mathcal{K}/R'}$ is torsion, it is easily seen that the R'-module $Ext^1_{R'}(\omega_{\mathcal{K}/R'}, R')$ is isomorphic to $\omega_{\mathcal{K}/R'}$.

We put $d = [K' : K]$. We denote by $\mathfrak{M}'$ the maximal ideal of R', and by μ the Galois group of the extension K'/K. We let μ act on K' from the left.

We put $X = \prod_{R'/R} \mathcal{G}'$ as before. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\ \alpha\ } & X^{\mu} \\
{\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\
\prod_{R'/R}(\mathcal{G} \times_R R') & \xrightarrow[\delta = \prod_{R'/R} h]{} & X
\end{array}
$$

where α is the isomorphism from Proposition 4.1, and β and γ are the tautological morphisms. It is easily seen that $Lie(\cdot)$ commutes with Weil restriction, so we have canonical isomorphisms of R-modules

$$
Lie\big(\prod_{R'/R}(\mathcal{G} \times_R R')\big) \quad \cong \quad Lie(\mathcal{G} \times_R R')
$$

$$
Lie(X) \quad \cong \quad Lie(\mathcal{G}')
$$

that identify δ with $Lie(h)$.

Arguing as in the proof of [13, 3.2] we see that $Lie(\gamma)$ is an isomorphism onto $Lie(\mathcal{G}')^{\mu}$, so that we can use $Lie(\gamma) \circ Lie(\alpha)$ to identify $Lie(\mathcal{G})$ with $Lie(\mathcal{G}')^{\mu}$. Also, taking formal parameters, it is easily seen that the morphism of R'-modules

$$
Lie(\mathcal{G}')^{\mu} \otimes_R R' \to Lie(\mathcal{G} \times_R R')
$$

obtained from $Lie(\beta)$ by extension of scalars is an isomorphism. Modulo this isomorphism, the morphism of R'-modules $Lie(h) : Lie(\mathcal{G} \times_R R') \to Lie(\mathcal{G}')$ corresponds to the inclusion

$$
Lie(\mathcal{G}')^{\mu} \otimes_R R' \to Lie(\mathcal{G}')
$$

Let $\{\overline{e}_1, \ldots, \overline{e}_m\}$ be a k-basis of $Lie(\mathcal{G}'_s)$ such that $\mu \cong \mu_d(k)$ acts on $\overline{e}_i$ by

$$
\overline{e}_i * \zeta = \zeta^{a_i} \cdot \overline{e}_i
$$

for each $\zeta \in \mu_d(k)$ and each $i \in \{1, \ldots, m\}$, with $a_i \in \{0, \ldots, d-1\}$. Then by Corollary 4.8, the occurring a_i (with multiplicities) are the K'-jumps of G (with multiplicities), so we have to show that there exists an isomorphism

$$
Lie(\mathcal{G}')/(Lie(\mathcal{G}')^{\mu} \otimes_R R') \cong \oplus_{i=1}^m R'_{a_i}
$$

We have a natural isomorphism of k-vector spaces

$$
Lie(X_s) \cong Lie(\mathcal{G}' \times_{R'} R'_d)
$$

Consider the projection

$$
Lie(\theta_0^d) : Lie(X_s) \to Lie(\mathcal{G}'_s)
$$

and lift the basis $\{\overline{e}_1, \ldots, \overline{e}_m\}$ of $Lie(\mathcal{G}'_s)$ to a tuple $\{e_1, \ldots, e_m\}$ of elements in $Lie(X_s)$ such that $\mu \cong \mu_d(k)$ acts on e_i by

$$
e_i * \zeta = \zeta^{a_i} \cdot e_i
$$

for each $\zeta \in \mu_d(k)$ and each $i \in \{1, \ldots, m\}$. Choose a uniformizer π' in R'. Since $\{e_1, \ldots, e_m\}$ is a R'_d-basis for $Lie(\mathcal{G}' \times_{R'} R'_d)$, we see that

$$
\{v_{i,j} = e_i \cdot (\pi')^j \,|\, i = 1, \ldots, m, \ j = 0, \ldots, d-1\}
$$

is a k-basis of $Lie(\mathcal{G}' \times_{R'} R'_d) \cong Lie(X_s)$, and $\zeta \in \mu_d(k)$ acts on $v_{i,j}$ by

$$
v_{i,j} * \zeta = \zeta^{a_i - j} \cdot v_{i,j}
$$

(see the construction of the group action in [13, 2.4] for the origin of the sign in the exponent $-j$).

If V_1 and V_2 are (right) $R[\mu]$-modules of finite type, free over R, and $\varphi : V_1 \otimes_R k \to V_2 \otimes_R k$ is an isomorphism of $k[\mu]$-modules, then φ lifts to an isomorphism of $R[\mu]$-modules $\varphi' : V_1 \to V_2$. To see this, take any morphism of R-modules $\phi : V_1 \to V_2$ lifting φ. The morphism

$$\varphi' : V_1 \to V_2 : v \mapsto \frac{1}{d} \sum_{\zeta \in \mu} \phi(v * \zeta) * \zeta^{-1}$$

is a morphism of $R[\mu]$-modules. It is an isomorphism since its reduction modulo π is the isomorphism φ.

This implies that we can lift the k-basis $\{v_{i,j}\}$ of $Lie(X_s)$ to a R-basis $\{w_{i,j}\}$ of $Lie(\mathcal{G}')$ such that the action of $\mu \cong \mu_d(k) \cong \mu_d(R)$ on $w_{i,j}$ is given by

$$w_{i,j} * \xi = \xi^{a_i - j} \cdot w_{i,j}$$

for each $\xi \in \mu_d(R)$ and all i, j. We see that the R-module

$$Lie(\mathcal{G}) = Lie(\mathcal{G}')^\mu$$

is generated by the elements w_{i,a_i} with $i = 1, \ldots, m$. We observe, moreover, that

$$\pi \cdot Lie(\mathcal{G}') \subset Lie(\mathcal{G}')^\mu \otimes_R R'$$

because $Lie(\mathcal{G}')^\mu \otimes_R R'$ is a sub-R'-module of $Lie(\mathcal{G}')$ of the same rank, and, for any v in $Lie(\mathcal{G}')$, we have $\pi \cdot v \in Lie(\mathcal{G}')^\mu$ iff $v \in Lie(\mathcal{G}')^\mu$.

Hence, we can conclude that

$$
\begin{aligned}
Lie(\mathcal{G}')/(Lie(\mathcal{G}')^\mu \otimes_R R') &\cong Lie(\mathcal{G}' \times_{R'} R'_d)/(\sum_{i=1}^m R'_d \cdot v_{i,a_i}) \\
&\cong Lie(\mathcal{G}' \times_{R'} R'_d)/(\sum_{i=1}^m (\pi')^{a_i} R'_d \cdot e_i) \\
&\cong \oplus_{i=1}^m R'_{a_i}
\end{aligned}
$$

$\square$

Lemma 4.11. *Let G be a smooth commutative algebraic K-group, and assume that $G \times_K K'$ admits a Néron model for every finite tame extension K' of K. For all $d, n \in \mathbb{N}'$ and each $i \in \{0, \ldots, d\}$ we have $F_d^i \mathcal{G}_s = F_{dn}^{in} \mathcal{G}_s$.*

Proof. For $i = 0$ the statement is obvious, so assume $i > 0$. Let $K' \subset K''$ be extensions of K of degree d, resp. dn, with rings of integers R', resp., R''. Denote by $\mathcal{G}'$ and $\mathcal{G}''$ the Néron models of $G \times_K K'$, resp. $G \times_K K''$. It suffices to show that the kernels of the truncation morphisms

$$
\begin{aligned}
\phi_1 &: \mathcal{G}_s \to Green_i(\mathcal{G}') \\
\phi_2 &: \mathcal{G}_s \to Green_{in}(\mathcal{G}'')
\end{aligned}
$$

coincide. We have a natural morphism $\mathcal{G}' \times_{R'} R'' \to \mathcal{G}''$ inducing a morphism $\mathcal{G}' \to \prod_{R''/R'} \mathcal{G}''$, which is a closed immersion by Proposition 4.1. Moreover, $Green_{in}(\mathcal{G}'')$ is canonically isomorphic to $Green_i(\prod_{R''/R'} \mathcal{G}'')$, and since Weil restriction respects closed immersions, we obtain a closed immersion of group k-schemes

$$\phi_{1,2} : Green_i(\mathcal{G}') \to Green_{in}(\mathcal{G}'')$$

such that $\phi_2 = \phi_{1,2} \circ \phi_1$. □

Definition 4.12. *Let G be a smooth commutative algebraic K-group, and assume that $G \times_K K'$ admits a Néron model for every finite tame extension of K. Denote by q the characteristic of k. For each element $\alpha = a/b$ of $\mathbb{Z}_{(q)} \cap [0, 1[$, with $a \in \mathbb{N}$ and $b \in \mathbb{N}'$, we put $\widetilde{F}^\alpha \mathcal{G}_s = F_b^a \mathcal{G}_s$. By Lemma 4.11, this definition does not depend on the choice of a and b. Then $\widetilde{F}^\bullet \mathcal{G}_s$ is a decreasing filtration on $\mathcal{G}_s$ by subgroup k-schemes.*

Let ρ be an element of $\mathbb{R} \cap [0, 1[$. We put $\widetilde{F}^{>\rho} \mathcal{G}_s = \widetilde{F}^\beta \mathcal{G}_s$, where β is any value in $\mathbb{Z}_{(q)} \cap \,]\rho, 1[$ such that $\widetilde{F}^{\beta'} \mathcal{G}_s = \widetilde{F}^\beta \mathcal{G}_s$ for all β' in $\mathbb{Z}_{(q)} \cap \,]\rho, \beta]$. If $\rho \neq 0$ we put $\widetilde{F}^{<\rho} \mathcal{G}_s = \widetilde{F}^\gamma \mathcal{G}_s$ where γ is any value in $\mathbb{Z}_{(q)} \cap [0, \rho[$ such that $\widetilde{F}^{\gamma'} \mathcal{G}_s = \widetilde{F}^\gamma \mathcal{G}_s$ for all γ' in $\mathbb{Z}_{(q)} \cap [\gamma, \rho[$. We put $\widetilde{F}^{<0} \mathcal{G}_s = \mathcal{G}_s$.

We define
$$\widetilde{Gr}^\rho \mathcal{G}_s = \widetilde{F}^{<\rho} \mathcal{G}_s / \widetilde{F}^{>\rho} \mathcal{G}_s$$
We say that $j \in \mathbb{R} \cap [0, 1[$ is a jump of G if $\widetilde{Gr}^j \mathcal{G}_s \neq 0$. The multiplicity of j is the dimension of $\widetilde{Gr}^j \mathcal{G}_s$. We say that the multiplicity of j as a jump of G is zero if j is not a jump of G.

It follows immediately from the definition that the sum of the multiplicities of the jumps of G equals the dimension of G. As noted by Edixhoven in [13, 5.4.5], it is not clear if the jumps of G are rational numbers, but one can be more precise if G is a tamely ramified semi-abelian variety over K.

Proposition 4.13. *Denote by q the characteristic of k. Assume that G is a semi-abelian K-variety, and that G acquires semi-abelian reduction over a tame finite extension K' of K of degree d. For any $\alpha \in \mathbb{Z}_{(q)} \cap [0, 1[$ we have*
$$\widetilde{F}^{<\alpha} \mathcal{G}_s = \widetilde{F}^\alpha \mathcal{G}_s = F_d^{\lceil \alpha \cdot d \rceil} \mathcal{G}_s$$
If we denote by $j_1(G, K'), \ldots, j_u(G, K')$ the K'-jumps of G, with multiplicities $m_1(G, K'), \ldots, m_u(G, K')$, then the jumps of G are given by
$$j_1(G, K')/d, \ldots, j_u(G, K')/d$$
with the same multiplicities. In particular, the jumps of G belong to $\mathbb{Z}[1/d] \cap [0, 1[$.

Moreover, if L/K is any finite tame extension of K, of degree e, then the set of L-jumps of G is
$$\{\lfloor j_1(G, K') \cdot e/d \rfloor, \ldots, \lfloor j_u(G, K') \cdot e/d \rfloor\}$$
The multiplicity of $j \in \{0, \ldots, e-1\}$ as a L-jump of G equals
$$\sum_{i \in T_j} m_i(G, K')$$
with $T_j = \{i \in \{1, \ldots, u\} \mid j = \lfloor j_i(G, K') \cdot e/d \rfloor\}$.

Proof. Write α as $i/(dn)$ with $d, n \in \mathbb{N}'$. We'll show that
$$(4.4) \qquad \widetilde{F}^\alpha \mathcal{G}_s := F_{dn}^i \mathcal{G}_s = F_d^{\lceil i/n \rceil} \mathcal{G}_s$$
The remainder of the statement follows from (4.4) by some elementary combinatorics.

So let us prove (4.4). Let $K' \subset K''$ be an extension of degree n. Denote by R' and R'' the ring of integers of K', resp. K'', and by $\mathcal{G}'$ and $\mathcal{G}''$ the Néron

models of $G \times_K K'$, resp. $G \times_K K''$. Since G has semi-abelian reduction over K', the natural morphism $\mathcal{G}' \times_{R'} R'' \to \mathcal{G}''$ is an open immersion, and $Lie(\mathcal{G}'_s)$ and $Lie(\mathcal{G}''_s)$ are isomorphic as k-vector spaces with $\mu'' = G(K''/K)$-action. In particular, $G(K''/K')$ acts trivially on $Lie(\mathcal{G}''_s)$. By Corollary 4.8, this means that $Gr^j_{nd}\mathcal{G}_s = 0$ if j is not divisible by n. It follows that

$$F^j_{dn}\mathcal{G}_s = F^{\lceil j/n \rceil \cdot n}_{dn}\mathcal{G}_s = F^{\lceil j/n \rceil}_d \mathcal{G}_s$$

for $j \in \{0, \ldots, dn\}$. $\qquad\square$

Remark. If G is the Jacobian of a smooth and proper K-curve with a K-rational point, then the jumps of G are rational, without any tameness condition. In fact, much more can be said: see [15, 8.4]. $\qquad\square$

4.2. **Chai's base change conductor.** Let G be a smooth commutative algebraic K-group that admits a Néron model $\mathcal{G}$. Let K' be a finite separable extension of K of degree d, and denote by R' the normalization of R in K'. Assume that $G' = G \times_K K'$ admits a Néron model $\mathcal{G}'$. By the universal property of the Néron model, there exists a unique morphism of group R'-schemes

$$h : \mathcal{G} \times_R R' \to \mathcal{G}'$$

that extends the canonical isomorphism between the generic fibers. Since $h_{K'}$ is an isomorphism, the map $Lie(h)$ injects $Lie(\mathcal{G} \times_R R')$ into $Lie(\mathcal{G}')$, and the quotient R'-module

$$Lie(\mathcal{G}')/(Lie(\mathcal{G} \times_R R'))$$

is a torsion module of finite type over R'. If we denote by $\mathfrak{M}'$ the maximal ideal of R' and if we put $R'_i = R'/(\mathfrak{M}')^i$ for each $i \in \mathbb{Z}_{>0}$, we get a decomposition

$$Lie(\mathcal{G}')/(Lie(\mathcal{G} \times_R R')) \cong \bigoplus_{i=1}^{v} R'_{c_i(G,K') \cdot d}$$

with $0 < c_1(G, K') \leq \ldots \leq c_v(G, K')$ in $\mathbb{Z}[1/d]$.

Definition 4.14. *We call the tuple of rational numbers* $(c_1(G, K'), \ldots, c_v(G, K'))$ *the tuple of K'-elementary divisors associated to G, and we call*

$$c(G, K') = c_1(G, K') + \cdots + c_v(G, K') = \frac{1}{d} \cdot \mathrm{length}_{R'} \left(Lie(\mathcal{G}')/(Lie(\mathcal{G} \times_R R')) \right)$$

the K'-base change conductor associated to G.

As a special case, we recall the following definition from [8, 2.4].

Definition 4.15 (Chai)**.** *Let A be a semi-abelian variety, and let K' be a finite separable extension of K such that $A \times_K K'$ has semi-abelian reduction. The values $c_i(A, K')$ and $c(A, K')$ only depend on A, and not on K'. We call them the elementary divisors, resp. the base change conductor of A.*

Note that our definition differs slightly from the one in [8, 2.4]. Chai extends the tuple of elementary divisors by adding zeroes to the left until the length of the tuple equals the dimension of A. Our definition is more convenient for the purpose of this paper.

Proposition 4.16. *For any semi-abelian variety A, we have $c(A) = 0$ iff A has semi-abelian reduction.*

Proof. The "if" part is obvious, so let us prove the converse implication. Take a finite separable extension K' of K such that $A' = A \times_K K'$ has semi-abelian reduction. Denote by R' the normalization of R in K' and by $\mathcal{A}$ and $\mathcal{A}'$ the Néron models of A, resp. A'. Consider the canonical morphism $h : \mathcal{A} \times_R R' \to \mathcal{A}'$. Then $c(A) = 0$ implies that $Lie(h)$ is an isomorphism. Hence, h is étale, and since h_K is an isomorphism, h is an open immersion [6, 2.3.2']. Therefore, A has semi-abelian reduction. $\qquad\square$

4.3. A comparison result.

Theorem 4.17. *Let K' be a finite tame extension of K, of degree d. Let G be a smooth commutative algebraic K-group, and assume that G and $G' = G \times_K K'$ admit Néron models. Let $j_1(G, K') < \ldots < j_u(G, K')$ be the non-zero K'-jumps of G, with respective multiplicities $m_1(G, K'), \ldots, m_u(G, K')$.*

Then the tuple of K'-elementary divisors $c_i(G, K')$ associated to G is

$$\left(\underbrace{j_1(G, K')/d, \ldots, j_1(G, K')/d}_{m_1(G,K')\times}, \ldots, \underbrace{j_u(G, K')/d, \ldots, j_u(G, K')/d}_{m_u(G,K')\times} \right)$$

and we have

$$d \cdot c(G, K') = \sum_{i=1}^{u} (m_i(G, K') \cdot j_i(G, K'))$$

Proof. This follows immediately from Theorem 4.10. $\qquad\square$

Combined with the fact that the sum of the multiplicities of the K'-jumps of G equals the dimension of G, Theorem 4.17 shows that the K'-jumps (with multiplicities) and the K'-elementary divisors determine each other.

Corollary 4.18. *Let A be a tamely ramified semi-abelian K-variety, and let $j_1(A) < \ldots < j_u(A)$ be the non-zero jumps associated to A, with respective multiplicities $m_1(A), \ldots, m_u(A)$. Then the tuple of elementary divisors of A is given by*

$$\left(\underbrace{j_1(A), \ldots, j_1(A)}_{m_1(A)\times}, \ldots, \underbrace{j_i(A), \ldots, j_i(A)}_{m_i(A)\times}, \ldots, \underbrace{j_u(A), \ldots, j_u(A)}_{m_u(A)\times} \right)$$

and we have

$$c(A) = \sum_{i=1}^{u} (m_i(A) \cdot j_i(A))$$

Proof. Apply Theorem 4.17 and Proposition 4.13. $\qquad\square$

Corollary 4.19. *If A is a tamely ramified semi-abelian K-variety, then its elementary divisors are contained in the interval $]0, 1[$.*

Corollary 4.20. *If A is a tamely ramified semi-abelian K-variety, then A has semi-abelian reduction iff 0 is the only jump of A.*

Corollary 4.21. *Let A be a tamely ramified semi-abelian K-variety, and let K' be a finite tame extension of K such that A acquires semi-abelian reduction over K'. Put $e = [K' : K]$ and $\mu = G(K'/K)$, and denote by $\mathcal{A}'$ the Néron model of*

$A \times_K K'$. Then for any $\zeta \in \mu \cong \mu_e(k)$, the determinant of the action of ζ on $Lie(\mathcal{A}'_s)$ equals $\zeta^{e \cdot c(A)}$.

Proof. This follows from Corollary 4.8, Proposition 4.13 and Corollary 4.18. $\square$

5. Jumps and monodromy eigenvalues

In this section, we assume that K is strictly henselian and that k is algebraically closed. Recall that for any real number x we denote by $[x]$ its decimal part $[x] = x - \lfloor x \rfloor \in [0, 1[$.

The following lemma describes the behaviour of the jumps of a tamely ramified semi-abelian K-variety under tame base change. For later use, we remark that the proof remains valid if K is strictly henselian but k is not necessarily perfect.

Lemma 5.1. *Let A be a semi-abelian K-variety, and assume that A acquires semi-abelian reduction on some finite tame extension L of K, of degree e. Let K' be a finite tame extension of K, of degree d. Let $j_1(A), \ldots, j_u(A)$ be the jumps of A, with respective multiplicities $m_1(A), \ldots, m_u(A)$. Then the set of jumps of $A' = A \times_K K'$ is*

$$J = \{[d \cdot j_1(A)], \ldots, [d \cdot j_u(A)]\} \subset \mathbb{Z}[1/e] \cap [0, 1[$$

Moreover, for any $j \in J$, the multiplicity of j as a jump of A' equals

$$\sum_{i \in S_j} m_i(A)$$

where S_j is the subset of $\{1, \ldots, u\}$ consisting of indices i such that $[d \cdot j_i(A)] = j$.

Proof. We may assume that K' is contained in L. Put $n = e/d$. Denote by $\mathcal{B}$ the Néron model of $B = A \times_K L$, and put $\mu = G(L/K) \cong \mu_e(k)$ and $\mu' = G(L/K') \cong \mu_n(k)$. For $a = 0, \ldots, e - 1$ we denote by $V[a]$ the subspace of $Lie(\mathcal{B}_s)$ where each $\zeta \in \mu$ acts by $v * \zeta = \zeta^a \cdot v$. Likewise, for $b = 0, \ldots, n - 1$ we denote by $V'[b]$ the subspace of $Lie(\mathcal{B}_s)$ where each $\xi \in \mu'$ acts by $v * \xi = \xi^b \cdot v$. Then we obviously have

$$V'[b] = \bigoplus_{a \equiv b \bmod n} V[a]$$

By Proposition 4.13 and Corollary 4.8 we know that a/e is a jump of A with multiplicity $m > 0$ iff $V[a]$ has dimension m, and likewise, b/n is a jump of A' with multiplicity $m' > 0$ iff $V'[b]$ has dimension m', so the result follows. $\square$

Lemma 5.2. *Let F be a perfect field, and let A be a semi-abelian F-variety. If B is a connected smooth subgroup F-scheme of A, then B is semi-abelian.*

Proof. By [9, 2.3] there exists a unique connected smooth linear subgroup L of B such that the quotient B/L is an abelian variety. By [2, XVII.7.2.1] we know that L is a product of a unipotent group U and a torus. But there are no non-trivial morphisms of F-groups from U to an abelian variety [9, 2.3] or to a torus [2, XVII.2.4], so there are no non-trivial morphisms from U to A. Therefore, U is trivial, and B is semi-abelian. $\square$

Lemma 5.3. *If A is a tamely ramified semi-abelian K-variety, with Néron model $\mathcal{A}$, then $\widetilde{F}^{>0}\mathcal{A}_s$ is unipotent, and $(\widetilde{Gr}^0 \mathcal{A}_s)^o$ semi-abelian.*

Proof. It follows immediately from Corollary 4.8 that $\widetilde{F}^{>0}\mathcal{A}_s$ is unipotent. Let K'/K be a finite tame extension such that the Néron model $\mathcal{A}'$ of $A' = A \times_K K'$ has semi-abelian reduction, and denote by μ the Galois group $G(K'/K)$. By Proposition 4.13 and Corollary 4.8 we have an isomorphism

$$\widetilde{Gr}^0 \mathcal{A}_s \cong (\mathcal{A}'_s)^\mu$$

This is a smooth subgroup scheme of $\mathcal{A}'_s$, and since $(\mathcal{A}'_s)^o$ is semi-abelian, $(\widetilde{Gr}^0 \mathcal{A}_s)^o$ is semi-abelian by Lemma 5.2. $\qquad\square$

Corollary 5.4. *If A is a tamely ramified semi-abelian K-variety, with Néron model $\mathcal{A}$, and $(c_1, \ldots, c_v)$ is its tuple of elementary divisors, then the length v of the tuple equals the unipotent rank of $\mathcal{A}_s^o$.*

Proof. By Corollary 4.18, the length v of the tuple of elementary divisors equals the sum of the multiplicities of the non-zero jumps of A, but this is precisely the dimension of $\widetilde{F}^{>0}\mathcal{A}_s$. $\qquad\square$

For any integer $i > 0$, we denote by $\Phi_i(t) \in \mathbb{Z}[T]$ the cyclotomic polynomial whose roots are the primitive i-th roots of unity. Its degree equals $\varphi(i)$, with $\varphi(\cdot)$ the Euler function. Recall that we fixed a topological generator σ of the tame monodromy group $G(K^t/K)$, and that we denote by $\tau : \mathbb{Q} \to \mathbb{Z}_{>0}$ the function which sends a rational number to its order in the group $\mathbb{Q}/\mathbb{Z}$.

Theorem 5.5. *Let A be a tamely ramified abelian K-variety, and let $j_1(A), \ldots, j_u(A)$ be the jumps of A, with respective multiplicities $m_1(A), \ldots, m_u(A)$. Denote by e the degree of the minimal extension L of K where A acquires semi-abelian reduction. For each divisor d of e we put*

$$
\begin{aligned}
J_d &= \{i \in \{1, \ldots, u\} \mid \tau(j_i(A)) = d\} \\
\nu_d &= 2 \cdot \left(\sum_{i \in J_d} m_i(A) \right)
\end{aligned}
$$

Then ν_d is divisible by $\varphi(d)$, and the characteristic polynomial $P_\sigma(t)$ of σ on $V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is given by

$$P_\sigma(t) = \prod_{d|e} \Phi_d(t)^{\nu_d/\varphi(d)}$$

Proof. The polynomial $P_\sigma(t)$ belongs to $\mathbb{Z}[t]$, and its zeroes are roots of unity whose orders divide e [3, IX.4.3]. Hence, $P_\sigma(t)$ is a product of cyclotomic polynomials $\Phi_d(t)$ with $d|e$. For each divisor d of e, we denote by r_d the number of zeroes of $P_\sigma(t)$ (counted with multiplicities) that are primitive d-th roots of unity. It suffices to show that $r_d = \nu_d$ for each divisor d of e. We proceed by induction on d.

By Lemma 5.3, the multiplicity of 0 as a jump of A equals the dimension of the semi-abelian part of $\mathcal{A}_s^o$. It is well-known that twice this dimension is equal to the multiplicity of 1 as an eigenvalue of σ on $V_\ell A$ (see for instance [19, 1.3]), so we find $r_1 = \nu_1$.

Now fix a divisor d of e and assume that $\nu_{d'} = r_{d'}$ for all divisors d' of d with $d' < d$. Note that the multiplicity m_d of 1 as a zero of $P_{\sigma^d}(t)$, the characteristic polynomial of σ^d on $V_\ell(A)$, equals $\sum_{d'|d} r_{d'}$. Applying the previous argument to

$A \times_K K(d)$, we see that m_d equals twice the multiplicity of 0 as a jump of $A \times_K K(d)$. By Lemma 5.1, we obtain

$$\sum_{d'|d} r_{d'} = m_d = \sum_{d'|d} \nu_{d'}$$

so by the induction hypothesis we see that $r_d = \nu_d$. □

Corollary 5.6. *Let A be a tamely ramified abelian K-variety. If A has a jump j with $\tau(j) = d$, then the primitive d-th roots of unity are monodromy eigenvalues of σ on $H^1(A \times_K K^t, \mathbb{Q}_\ell)$.*

In fact, we can give a more precise description of the relation between jumps and monodromy. First, we need some auxiliary lemmas.

Lemma 5.7. *For each $n \in \mathbb{Z}_{>0}$ there exists a $\mathbb{Q}$-rational representation ρ of $\mathbb{Z}/n\mathbb{Z}$ such that $\rho(1)$ has characteristic polynomial $\Phi_n(t)$.*

Proof. We proceed by induction on n. For $n = 1$ the result is clear, so assume that $n > 1$ and that the lemma holds for all $n' < n$. This implies that, for each divisor m of n with $m < n$, there exists a $\mathbb{Q}$-rational representation ρ_m of $\mathbb{Z}/n\mathbb{Z}$ such that $\rho_m(1)$ has characteristic polynomial $\Phi_m(t)$. Let ρ' be a complex representation of $\mathbb{Z}/n\mathbb{Z}$ such that $\rho'(1)$ has characteristic polynomial $\Phi_n(t)$. We'll show that ρ' is defined over $\mathbb{Q}$. By [36, §12.1], it is enough to prove that the character $\chi_{\rho'}$ of ρ' belongs to the representation ring $R_\mathbb{Q}(\mathbb{Z}/n\mathbb{Z})$. If we denote by ρ_{reg} the regular representation of $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Q}$, then

$$\rho_{reg} \otimes_\mathbb{Q} \mathbb{C} \cong \rho' \oplus \left(\bigoplus_{m|n,\, m<n} (\rho_m \otimes_\mathbb{Q} \mathbb{C}) \right)$$

Hence,

$$\chi_{\rho'} = \chi_{\rho_{reg}} - \sum_{m|n,\, m<n} \chi_{\rho_m}$$

belongs to $R_\mathbb{Q}(\mathbb{Z}/n\mathbb{Z})$. □

Lemma 5.8. *Let A be an abelian K-variety of dimension g, and let γ be an element of $G(K^s/K)$. For any $i \in \{0, \ldots, 2g\}$, the characteristic polynomial $P_\gamma^{(i)}(t)$ of γ on $H^i(A \times_K K^s, \mathbb{Q}_\ell)$ belongs to $\mathbb{Z}[t]$. It is independent of ℓ, and it is a product of cyclotomic polynomials.*

Proof. If $i = 1$ then this follows from [3, IX.4.3]. Hence, by Lemma 5.7, there exists an automorphism of finite order γ' of the $\mathbb{Q}$-vector space $V = \mathbb{Q}^{2g}$ such that the characteristic polynomial of γ' on V is $P_\gamma^{(1)}(t)$. For any $i \geq 0$ there exists a $G(K^s/K)$-equivariant isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$H^i(A \times_K K^s, \mathbb{Q}_\ell) \cong \bigwedge^i H^1(A \times_K K^s, \mathbb{Q}_\ell)$$

so that $P_\gamma^{(i)}(t)$ coincides with the characteristic polynomial of $\wedge^i \gamma'$ on $\bigwedge^i V$. It follows that $P_\gamma^{(i)}(t)$ is a product of cyclotomic polynomials, independent of ℓ. □

Definition 5.9. *We say that a polynomial $Q(t)$ in $\mathbb{Z}[t]$ is p-tame if it is of the form*

$$Q(t) = \Phi_{n_1}(t) \cdot \ldots \cdot \Phi_{n_j}(t)$$

with $j \in \mathbb{Z}_{>0}$ and $n_1, \ldots, n_j \in \mathbb{N}'$.

Lemma 5.10. *Consider the unique ring morphism*

$$\mathbb{Z}[t] \to k[t]$$

mapping t to t.

    (1) *Let ζ be a primitive d-th root of unity in k, with $d \in \mathbb{N}'$. Fix an algebraic closure $\mathbb{Q}^a$ of $\mathbb{Q}$ and a primitive d-th root of unity ξ in $\mathbb{Q}^a$. Let n be an element of $\mathbb{Z}_{>0}$ and let a be a tuple in $\mathbb{N}^n$. If $Q(t)$ is a p-tame polynomial in $\mathbb{Z}[t]$ whose image in $k[t]$ is divisible by $\prod_{i=1}^{n}(t - \zeta^{a_i})$, then $Q(t)$ is divisible by $\prod_{i=1}^{n}(t - \xi^{a_i})$ in $\mathbb{Q}^a[t]$.*

    (2) *If $Q_1(t)$ and $Q_2(t)$ are p-tame polynomials in $\mathbb{Z}[t]$ whose images in $k[t]$ coincide, then $Q_1(t) = Q_2(t)$.*

Proof. (1) If k has characteristic zero, the result follows by considering the embedding of $\mathbb{Q}(\xi)$ in k that maps ξ to ζ. So assume that $p > 1$. Denote by $W(k)$ the ring of k-Witt vectors, and consider the unique embedding of $\mathbb{Z}[\xi]$ in $W(k)$ such that the image of ξ in the residue field k of $W(k)$ equals ζ. We may assume that the roots of $Q(t)$ are d-th roots of unity. The result then follows from the fact that the reduction map

$$\mu_d(W(k)) \to \mu_d(k)$$

is a bijection.

    (2) Any p-tame polynomial $Q(t)$ in $\mathbb{Z}[t]$ divides $(t^u - 1)^v$ for some $u \in \mathbb{N}'$ and some $v \in \mathbb{N}$. Hence, the roots of its image in $k[t]$ are u-th roots of unity, so that (2) follows from (1). $\qquad\square$

Proposition 5.11. *Let G be a semi-abelian k-variety of dimension g, and let φ be an automorphism of G of order $d \in \mathbb{N}'$. We fix an algebraic closure $\mathbb{Q}_\ell^a$ of $\mathbb{Q}_\ell$. We fix a primitive d-th root of unity ξ in $\mathbb{Q}_\ell^a$, and a primitive d-th root of unity ζ in k. Let $a_1, \ldots, a_g$ be elements of $\{0, \ldots, d-1\}$ such that*

$$\zeta^{a_1}, \ldots, \zeta^{a_g}$$

are the eigenvalues of φ on $\mathrm{Lie}(G)$. Then the characteristic polynomial $Q_\varphi(t)$ of φ on $V_\ell G = T_\ell G \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is divisible by

$$\prod_{i=1}^{g}(t - \xi^{a_i})$$

Proof. We consider the Chevalley decomposition

$$0 \to T \to G \to B \to 0$$

of G, with T a k-torus and B an abelian k-variety. The action of φ on G induces automorphisms on T and B, which we denote again by φ. There exist φ-equivariant isomorphisms

$$\begin{aligned}
\mathrm{Lie}(G) &\cong \mathrm{Lie}(T) \oplus \mathrm{Lie}(B) \\
V_\ell G &\cong V_\ell T \oplus V_\ell B
\end{aligned}$$

so that it is enough to consider the case where $G = T$ or $G = B$.

    Case 1: $G = T$. If we denote by $X(T)$ the character group of T, then there are canonical isomorphisms

$$\begin{aligned}
(5.1) \qquad\qquad \mathrm{Lie}(T) &= \mathrm{Hom}_{\mathbb{Z}}(X(T), k) \\
(5.2) \qquad\qquad V_\ell T &= \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Q}_\ell(1))
\end{aligned}$$

The isomorphism (5.2) implies that $Q_\varphi(t)$ belongs to $\mathbb{Z}[t]$. Since φ has order $d \in \mathbb{N}'$, we see that $Q_\varphi(t)$ is a p-tame polynomial. Combining isomorphisms (5.1) and (5.2), we obtain that the image of $Q_\varphi(t)$ in $k[t]$ equals

$$\prod_{i=1}^{g}(t - \zeta^{a_i})$$

By Lemma 5.10, we may conclude that

$$Q_\varphi(t) = \prod_{i=1}^{g}(t - \xi^{a_i})$$

Case 2: $G = B$. Since ℓ-adic cohomology is a Weil cohomology, $Q_\varphi(t)$ coincides with the characteristic polynomial of φ on B (see for instance the appendix to [23]). In particular, $Q_\varphi(t)$ belongs to $\mathbb{Z}[t]$. Since the order of φ belongs to $\mathbb{N}'$, we see that $Q_\varphi(t)$ is a p-tame polynomial.

Since B is an abelian variety, its Hodge-de Rham spectral sequence degenerates at E_1 [32, 5.1]. This yields a natural short exact sequence

$$(5.3) \qquad 0 \to H^0(B, \Omega_B^1) \to H_{dR}^1(B) \to H^1(B, \mathcal{O}_B) \to 0$$

with $H_{dR}^1(B)$ the degree one de Rham cohomology of B. We have natural isomorphisms

$$(5.4) \qquad H^0(B, \Omega_B^1) \cong \omega_{B/k} \cong Lie(B)^\vee$$

First, assume that k has characteristic zero. Then de Rham cohomology is a Weil cohomology, so that $Q_\varphi(t)$ is also the characteristic polynomial of φ on $H_{dR}^1(B)$. By (5.3), (5.4) and Lemma 5.10, we see that $Q_\varphi(t)$ is divisible by

$$\prod_{i=1}^{g}(t - \xi^{a_i})$$

Now, assume that k has characteristic $p > 1$. If we denote by $H_{crys}^i(B)$ the degree i integral crystalline cohomology of B, then $H_{crys}^i(B)$ is a free $W(k)$-module for each i [18, II.7.1], so that there is a canonical isomorphism

$$(5.5) \qquad H_{dR}^1(B) \cong H_{crys}^1(B) \otimes_{W(k)} k$$

by [18, II.4.9.1]. But crystalline cohomology is a Weil cohomology too, which implies that the characteristic polynomial of φ on $H_{crys}^1(B)$ coincides with $Q_\varphi(t)$. By (5.3), (5.4) and (5.5), the image of $Q_\varphi(t)$ in $k[t]$ is divisible by

$$\prod_{i=1}^{g}(t - \zeta^{a_i})$$

By Lemma 5.10, we see that $Q_\varphi(t)$ is divisible by

$$\prod_{i=1}^{g}(t - \xi^{a_i})$$

in $\mathbb{Q}_\ell^a[t]$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Theorem 5.12. *Let A be a tamely ramified abelian K-variety, and denote by e the degree of the minimal extension of K where A acquires semi-abelian reduction. We fix an algebraic closure $\mathbb{Q}_\ell^a$ of $\mathbb{Q}_\ell$ and a primitive e-th root of unity ξ in $\mathbb{Q}_\ell^a$. If we denote by $j_1(A), \ldots, j_u(A)$ the jumps of A, with respective multiplicities $m_1(A), \ldots, m_u(A)$, then*

$$\prod_{i=1}^{u}(t - \xi^{e \cdot j_i(A)})^{m_i(A)}$$

divides the characteristic polynomial $P_\sigma(t)$ of the action of σ on $V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Proof. We put $\mu = G(K(e)/K)$. We denote by $(T_\ell A)^{ef}$ the essentially fixed part of the Tate module $T_\ell A$ [3, IX.4.1]. It is a sub-$\mathbb{Z}_\ell$-module of $T_\ell A$, closed under the $G(K^s/K)$-action. The action of $G(K^s/K)$ on $(T_\ell A)^{ef}$ factors through μ, and we have a canonical μ-equivariant isomorphism

$$(T_\ell A)^{ef} \cong T_\ell(\mathcal{A}(e)_s^o)$$

by [3, IX.4.2.6].

If we denote by ζ the image of σ in $\mu \cong \mu_e(k)$, then the characteristic polynomial of σ on $Lie(\mathcal{A}(e)_s^o)$ equals

$$\prod_{i=1}^{u}(t - \zeta^{e \cdot j_i(A)})^{m_i(A)}$$

by Corollary 4.8 and Proposition 4.13. The result now follows from Proposition 5.11. $\square$

Corollary 5.13. *Let A be a tamely ramified abelian K-variety of dimension g. The cyclotomic polynomial $\Phi_{\tau(c(A))}(t)$ divides the characteristic polynomial of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$.*

Proof. We keep the notations of Theorem 5.12. By Corollary 4.18 we have

$$c(A) = \sum_{i=1}^{u} m_i(A) \cdot j_i(A)$$

By Lemma 5.8 it suffices to show that

$$\xi^{e \cdot c(A)} = \prod_{i=1}^{u}(\xi^{e \cdot j_i(A)})^{m_i(A)}$$

is an eigenvalue of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$. However, by Theorem 5.12, this is a product of $\sum_{i=1}^{u} m_i(A) = g$ distinct entries in the sequence of roots (with multiplicities) of $P_\sigma(T)$, and hence an eigenvalue of σ on

$$H^g(A \times_K K^t, \mathbb{Q}_\ell) \cong \bigwedge^{g} H^1(A \times_K K^t, \mathbb{Q}_\ell)$$

$\square$

Corollary 5.14. *Let A be a tamely ramified abelian K-variety. We use the notations of Theorem 5.5. For every jump j of A, the multiplicity m of j satisfies*

$$m \leq \nu_{\tau(j)}/\varphi(\tau(j))$$

Proof. Compare Theorems 5.5 and 5.12. $\square$

6. Néron models and tame base change

In this section, we assume that K is strictly henselian and that k is algebraically closed. We adopt the following notation: if G is a smooth commutative algebraic K-group such that G admits a Néron model, then we denote this Néron model by $\mathcal{G}$. If $d \in \mathbb{N}'$ and $G(d) = G \times_K K(d)$ admits a Néron model, we denote it by $\mathcal{G}(d)$. If G is semi-abelian, we will often use the notations A and $\mathcal{A}$ instead of G and $\mathcal{G}$.

Definition 6.1. *Denote by AV the set of isomorphism classes of abelian k-varieties. Let G be a smooth commutative algebraic K-group such that $G(d)$ admits a Néron model $\mathcal{G}(d)$ for each $d \in \mathbb{N}'$. We denote by $u_G(d)$, $t_G(d)$ and $a_G(d)$ the unipotent, resp. reductive, resp. abelian rank of $\mathcal{G}(d)^o_s$. We put $\phi_G(d) = \phi(G(d))$ (see Definition 3.6) and we denote by $B_G(d) \in AV$ the isomorphism class of the abelian quotient in the Chevalley decomposition of $\mathcal{G}(d)^o_s$.*

Proposition 6.2. *Let A be a tamely ramified semi-abelian K-variety, and let $j_1(A), \ldots, j_u(A)$ be its jumps, with multiplicities $m_1(A), \ldots, m_u(A)$. Denote by e the degree of the minimal extension of K where A acquires semi-abelian reduction. Then for each $d \in \mathbb{N}'$ we have*

$$u_A(d) = \sum_{d \cdot j_i(A) \notin \mathbb{Z}} m_i(A)$$

and this value is also equal to the number of elementary divisors of $A(d)$. Moreover, the values $u_A(d)$, $t_A(d)$, $a_A(d)$ and $B_A(d)$ only depend on $d \bmod e$.

Proof. The expressions for $u_A(d)$ follow from Corollaries 4.18 and 5.4 and Lemma 5.1. Since $e \cdot j_i(A)$ belongs to $\mathbb{Z}$ for each $i \in \{1, \ldots, u\}$, by Proposition 4.13, the property $d \cdot j_i(A) \in \mathbb{Z}$ only depends on $d \bmod e$.

By the equality

$$\dim A = u_A(d) + t_A(d) + a_A(d)$$

the only thing left to show is that $B_A(d) \in AV$ only depends on $d \bmod e$. By Lemma 5.3, it is enough to show that $(Gr^0\mathcal{A}(d)_s)^o$ and $(Gr^0\mathcal{A}(d+e')_s)^o$ are isomorphic if $d \in \mathbb{N}'$ and e' is a multiple of e such that $d + e' \in \mathbb{N}'$. We put

$$\begin{aligned}
m &= d(d+e')e' \\
\mu_1 &= G(K(m)/K(d)) \\
\mu_2 &= G(K(m)/K(d+e')) \\
\mu_3 &= G(K(m)/K(e'))
\end{aligned}$$

By Corollary 4.8 we have

$$\begin{aligned}
(Gr^0\mathcal{A}(d)_s)^o &\cong ((\mathcal{A}(m)^o_s)^{\mu_1})^o \\
(Gr^0\mathcal{A}(d+e')_s)^o &\cong ((\mathcal{A}(m)^o_s)^{\mu_2})^o
\end{aligned}$$

Let ζ be a generator of $G(K(m)/K)$. Then ζ^d and $\zeta^{d+e'}$ generate μ_1, resp. μ_2. Since $A(e')$ has semi-abelian reduction, the natural morphism $\mathcal{A}(e')^o \times_{R(e')} R(m) \to \mathcal{A}(m)^o$ is an isomorphism. Hence, the action of $\mu_3 = <\zeta^{e'}>$ on $\mathcal{A}(m)^o_s$ is trivial, so the actions of ζ^d and $\zeta^{d+e'}$ coincide. $\qquad\qquad\square$

7. The order function

In this section, we assume that K is strictly henselian.

Definition 7.1. *Let G be a smooth commutative algebraic K-group of dimension g, and assume that G admits a Néron model $\mathcal{G}$. A distinguished gauge form on G is a degree g differential form ω of the form $\omega = j^*\omega'$, where $j : G \to \mathcal{G}$ is the natural open immersion of G into its Néron model, and ω' is a translation-invariant generator of the free rank 1 module $\Omega^g_{\mathcal{G}/R}$.*

Such a distinguished gauge form ω always exists [6, 4.2.3]. It is unique up to multiplication with an element in the unit group R^*, and it is translation-invariant w.r.t. the group multiplication on G.

Definition 7.2. *Let G be a smooth commutative algebraic K-group, and assume that $G(d)$ admits a Néron model $\mathcal{G}(d)$ for each $d \in \mathbb{N}'$. Let ω be a distinguished gauge form on G. For each $d \in \mathbb{N}'$, we put*

$$ord_G(d) = -ord(\omega(d))(e_{\mathcal{G}(d)}) = -ord_{\mathcal{G}(d)^o_s}(\omega(d))$$

We call $ord_G(\cdot)$ the order function of G.

This definition does not depend on the choice of distinguished gauge form. The equality

$$ord(\omega(d))(e_{\mathcal{G}(d)}) = ord_{\mathcal{G}(d)^o_s}(\omega(d))$$

follows from Proposition 2.2. The value $ord_G(d)$ measures the difference between $\mathcal{G}(d)$ and $\mathcal{G} \times_R R(d)$.

Proposition 7.3. *Let G be a smooth commutative algebraic K-group, and let d be an element of $\mathbb{N}'$ such that $G(d)$ admits a Néron model $\mathcal{G}(d)$. If ω is a gauge form on G, then*

$$ord_C(\omega(d)) = ord_{\mathcal{G}(d)^o_s}(\omega(d))$$

for each connected component C of $\mathcal{G}(d)_s$.

Proof. This follows immediately from the translation-invariance of ω. $\square$

Proposition 7.4. *Let G be a smooth commutative algebraic K-group, and assume that $G(d)$ admits a Néron model $\mathcal{G}(d)$ for each $d \in \mathbb{N}'$.*

Let d be an element of $\mathbb{N}'$, denote by h the canonical morphism $\mathcal{G} \times_R R(d) \to \mathcal{G}(d)$, and denote by $\mathcal{K}(d)$ the kernel of h. Then

$$ord_G(d) = \text{length}_{R(d)}\omega_{\mathcal{K}(d)/R(d)}$$

Proof. By the exact sequence of $R(d)$-modules

$$0 \longrightarrow \omega_{\mathcal{G}(d)/R(d)} \xrightarrow{\ \alpha\ } \omega_{\mathcal{G} \times_R R(d)/R(d)} \longrightarrow \omega_{\mathcal{K}(d)/R(d)} \longrightarrow 0$$

from Theorem 4.10, we have

$$\text{length}_{R(d)}\omega_{\mathcal{K}(d)/R(d)} = \text{length}_{R(d)}coker(\alpha)$$

Since α is an injective morphism of free $R(d)$-modules of the same rank,

$$\text{length}_{R(d)}coker(\alpha) = \text{length}_{R(d)}coker(det(\alpha))$$

But $det(\alpha)$ is nothing but the morphism of free rank one $R(d)$-modules

$$(7.1) \qquad det(\alpha) : e^*_{\mathcal{G}(d)}\Omega^g_{\mathcal{G}(d)/R(d)} \to e^*_{\mathcal{G} \times_R R(d)}\Omega^g_{\mathcal{G} \times_R R(d)/R(d)}$$

If we denote by $\omega(d)'$ the unique extension of $\omega(d)$ to a relative gauge form on $\mathcal{G} \times_R R(d)$, then the target of (7.1) is generated by the pull-back of $\omega(d)'$. Hence,

$$\text{length}_{R(d)}\omega_{\mathcal{K}(d)/R(d)} = -ord(\omega(d))(e_{\mathcal{G}(d)}) = ord_G(d)$$

$\square$

Proposition 7.5. *Let G be a smooth commutative algebraic K-group, and assume that $G(d)$ admits a Néron model $\mathcal{G}(d)$ for each $d \in \mathbb{N}'$. If we denote by*

$$j_1(G, K(d)), \ldots, j_u(G, K(d))$$

the $K(d)$-jumps of G, with respective multiplicities $m_1(G, K(d)), \ldots, m_u(G, K(d))$, then we have

$$ord_G(d) = c(G, K(d)) \cdot d = \sum_{i=1}^{u} m_i(G, K(d)) \cdot j_i(G, K(d))$$

for each $d \in \mathbb{N}'$.

In particular, if A is a tamely ramified semi-abelian K-variety, with jumps $j_1(A), \ldots, j_v(A)$ with respective multiplicities $m_1(A), \ldots, m_v(A)$, then we have

$$ord_A(d) = \sum_{i=1}^{v} m_i(A) \cdot \lfloor j_i(A) \cdot d \rfloor$$

for every $d \in \mathbb{N}'$.

Proof. Combine Theorem 4.10, Proposition 4.13 and Proposition 7.4. $\square$

Corollary 7.6. *Let A be a tamely ramified semi-abelian K-variety, and denote by e the degree of the minimal extension of K where A acquires semi-abelian reduction. For all $d \in \mathbb{N}'$ and all $q \in \mathbb{Z}_{>0}$ such that $d + q \cdot e \in \mathbb{N}'$, we have*

$$ord_A(d + q \cdot e) = ord_A(d) + q \cdot c(A) \cdot e$$

Moreover, we have

$$ord_A(d) \leq c(A) \cdot d$$

for all $d \in \mathbb{N}'$, with equality iff $e|d$.

Proof. The first assertion follows from the fact that the jumps of A belong to $(1/e)\mathbb{Z}$ (Proposition 4.13). For the second, note that with the notations of Proposition 7.5, we have

$$\begin{aligned}
ord_A(d) &= \sum_{i=1}^{v} m_i(A) \cdot \lfloor j_i(A) \cdot d \rfloor \\
&\leq \sum_{i=1}^{v} m_i(A) \cdot j_i(A) \cdot d \\
&= c(A) \cdot d \quad \text{(Proposition 4.18)}
\end{aligned}$$

with equality iff $j_i(A) \cdot d \in \mathbb{Z}$ for all i. By Lemma 5.1, $j_i(A) \cdot d \in \mathbb{Z}$ for all i iff all the jumps of $A(d)$ are zero, i.e., iff $A(d)$ has semi-abelian reduction (Corollary 4.20), i.e., iff $e|d$. $\square$

8. The motivic zeta function of an abelian variety

Throughout this section, we assume that K is complete and k algebraically closed. We keep the notations of Sections 6 and 7, in particular the ones introduced in Definition 6.1.

8.1. The motivic zeta function.

Definition 8.1. *Let G be a smooth commutative algebraic K-group of dimension g such that G admits a Néron model. Let ω be a distinguished gauge form on G. We define the motivic zeta function $Z_G(T)$ of G by*

$$Z_G(T) = \mathbb{L}^g \cdot \sum_{d \in \mathbb{N}'} \left(\int_{G(d)^b} |\omega(d)| \right) T^d \quad \in \mathcal{M}_k[[T]]$$

where $G(d)^b$ denotes the bounded part of $G(d)$ (Definition 3.9).

In particular, if k has characteristic zero and G is proper, we have

$$Z_G(T) = \mathbb{L}^g \cdot S(G, \omega; T) \in \mathcal{M}_k[[T]]$$

with $S(G, \omega; T)$ the motivic generating series associated to (G, ω) (Section 2.4).

Since ω is unique up to multiplication with a unit in R^*, this definition is independent of ω. Note that $G(d)$ admits a Néron model $\mathcal{G}(d)$ for each $d \in \mathbb{N}'$ because R is excellent (see Lemma 3.2 and Proposition 3.10).

Proposition 8.2. *Let G be a smooth commutative algebraic K-group of dimension g such that G admits a Néron model $\mathcal{G}$. Let ω be a distinguished gauge form on G. Then*

$$
\begin{aligned}
Z_G(T) &= \sum_{d \in \mathbb{N}'} [\mathcal{G}(d)_s] \mathbb{L}^{ord_G(d)} T^d \\
&= \sum_{d \in \mathbb{N}'} \left(\phi_G(d) \cdot (\mathbb{L} - 1)^{t_G(d)} \cdot \mathbb{L}^{u_G(d) + ord_G(d)} \cdot [B_G(d)] \cdot T^d \right)
\end{aligned}
$$

in $\mathcal{M}_k[[T]]$.

Proof. Since the formal π-adic completion of $\mathcal{G}(d)$ is a weak Néron model for $G(d)^b$, the first equality follows from Proposition 2.3 and Proposition 7.3. The second one follows from [25, 2.1]. $\square$

8.2. Cohomological interpretation.

Theorem 8.3. *Let A be a tamely ramified abelian K-variety. Denote by Add_A the set of elements d in $\mathbb{N}'$ such that $A(d)$ has purely additive reduction.*
Then

$$
\begin{aligned}
\chi_{top}(Z_A(T)) &= \sum_{d \in \mathbb{N}'} \chi_{top}(\mathcal{A}(d)_s) T^d \\
&= \sum_{d \in Add_A} \phi_A(d) T^d \\
&= \sum_{d \in \mathbb{N}'} \sum_{i \geq 0} (-1)^i Trace(\sigma^d \mid H^i(A \times_K K^t, \mathbb{Q}_\ell)) T^d
\end{aligned}
$$

in $\mathbb{Z}[[T]]$.

Proof. The first two equalities follow from Proposition 8.2, without assuming that A is tamely ramified. The last equality follows from the trace formula [25, 2.5+8]. □

8.3. **Proof of the monodromy conjecture for tamely ramified abelian varieties.**

Definition 8.4. *Let A be a semi-abelian K-variety, and take a finite extension K' of K such that $A' = A \times_K K'$ has semi-abelian reduction. We denote by $\mathcal{A}'$ the Néron model of A', and we define the potential toric rank $t_{pot}(A)$ of A to be the reductive rank of $(\mathcal{A}'_s)^o$. It is independent of K'.*

The notion of pole of a rational series in $\mathcal{M}_k[[\mathbb{L}^{-s}]]$ was defined in [34, §4] (here s is a formal variable). This notion requires some care because $\mathcal{M}_k$ might not be a domain. The following theorem is the main result of the present paper.

Theorem 8.5 (Monodromy conjecture for abelian varieties)**.** *Let A be a tamely ramified abelian K-variety of dimension g.*

(1) *The motivic zeta function $Z_A(T)$ is rational, and belongs to the subring*

$$\mathscr{R}_k^{c(A)} = \mathcal{M}_k \left[T, \frac{1}{1 - \mathbb{L}^a T^b} \right]_{(a,b) \in \mathbb{N} \times \mathbb{N}_0,\, a/b = c(A)}$$

of $\mathcal{M}_k[[T]]$. The zeta function $Z_A(\mathbb{L}^{-s})$ has a unique pole at $s = c(A)$, whose order is equal to $t_{pot}(A) + 1$. The degree of $Z_A(T)$ is equal to zero if $p = 1$ and A has potential good reduction, and strictly negative in all other cases.

(2) *The cyclotomic polynomial $\Phi_{\tau(c(A))}(t)$ divides the characteristic polynomial of the tame monodromy operator σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$. Hence, for every embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, the value $\exp(2\pi c(A)i)$ is an eigenvalue of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$.*

Proof. Statement (2) follows immediately from Corollary 5.13, so it suffices to prove (1). In order to detect poles of $Z_A(T)$, we will specialize $Z_A(T)$ by means of a ring morphism

$$\rho : \mathcal{M}_k \to F$$

such that F is a field. We write $c(A)$ as a_0/b_0, with $a_0 \in \mathbb{N}$ and $b_0 \in \mathbb{Z}_{>0}$. We assume that the image $\rho(\mathbb{L})$ of $\mathbb{L}$ in F is not a root of unity, and that $\rho(\mathbb{L})$ has a b_0-th root $\rho(\mathbb{L})_{b_0}$ in F. If $S(T)$ is an element of $\mathscr{R}_k^{c(A)}$ such that

$$\rho(S(T)) \in F[[T]]$$

has a pole of order n at $(\rho(\mathbb{L})_{b_0})^{-a_0}$, then it follows easily from the definition in [34, §4] that $S(\mathbb{L}^{-s})$ has a pole at $s = c(A)$ of order at least n.

The specialization morphism we'll use is the *Poincaré polynomial*

$$P_k : \mathcal{M}_k \to \mathbb{Z}[u, u^{-1}]$$

(see [24, §8]). If X is a k-variety of dimension d with r irreducible components of maximal dimension, then $P_k([X])$ is a polynomial in $\mathbb{Z}[u]$ of degree $2d$ whose leading coefficient equals r [24, 8.7]. In particular, this leading coefficient is strictly positive. The element $P_k(\mathbb{L})$ is equal to u^2. We denote by $\mathscr{F}$ the quotient field of $\cup_{n>0} \mathbb{Z}[u^{1/n}, u^{-1/n}]$.

Let e be the degree of the minimal extension of K where A acquires semi-abelian reduction. For each $\alpha \in \{1, \ldots, e\}$, we put

$$Z_A^{(\alpha)}(T) = \sum_{d \in \mathbb{N}' \cap (\alpha + \mathbb{N}e)} \left(\phi_A(d) \cdot (\mathbb{L} - 1)^{t_A(d)} \cdot \mathbb{L}^{u_A(d) + ord_A(d)} \cdot [B_A(d)] \cdot T^d \right)$$

By Proposition 8.2, we have

$$Z_A(T) = \sum_{\alpha=1}^{e} Z_A^{(\alpha)}(T)$$

Hence, it suffices to prove the following claims.

Claim 1. For each $\alpha \in \{1, \ldots, e\}$, the series $Z_A^{(\alpha)}(T)$ belongs to the sub-$\mathcal{M}_k[T]$-module $\mathcal{M}^\alpha$ of $\mathscr{R}_k^{c(A)}$ generated by the elements

$$(1 - \mathbb{L}^a T^b)^{-t_A(\alpha)-1}$$

with $(a, b) \in \mathbb{N} \times \mathbb{N}_0$, $a/b = c(A)$. Its specialization $P_k(Z_A^{(\alpha)}(T))$ in $\mathscr{F}[[T]]$ has a pole of order $t_A(\alpha) + 1$ at $T = u^{-2c(A)}$. The residue of this pole belongs to the subring $\cup_{n>0} \mathbb{Q}[u^{1/n}, u^{-1/n}]$ of $\mathscr{F}$, and its leading coefficient has sign $(-1)^{t_A(\alpha)+1}$.

Claim 2. The degree of $Z_A^{(\alpha)}(T)$ is zero if $p = 1$, $\alpha = e$ and $A(e)$ has good reduction, and strictly negative in all other cases.

First, we prove Claim 1. For each $\alpha \in \{1, \ldots, e\}$, we put $\alpha' = gcd(\alpha, e)$. It follows from [16, 5.7] that

$$\phi_A(d) = (d/\alpha')^{t_A(\alpha')} \phi_A(\alpha')$$

for every element d of $\mathbb{N}' \cap (\alpha + \mathbb{N}e)$. Moreover, $t_A(\alpha') = t_A(\alpha)$ by [16, 4.2]. Using Proposition 6.2 and Corollary 7.6, we can write

$$Z_A^{(\alpha)}(T) = \phi_A(\alpha')(\mathbb{L} - 1)^{t_A(\alpha)} \mathbb{L}^{u_A(\alpha) + ord_A(\alpha)} [B_A(\alpha)](\alpha')^{-t_A(\alpha)} S_A^{(\alpha)}(T)$$

with

$$S_A^{(\alpha)}(T) = \sum_{\{q \in \mathbb{N} \mid qe + \alpha \in \mathbb{N}'\}} (qe + \alpha)^{t_A(\alpha)} \mathbb{L}^{qc(A)e} T^{qe+\alpha}$$

We denote by n_α the smallest element of $\alpha + \mathbb{N}e$ that is divisible by p, and we put $q_\alpha = (n_\alpha - \alpha)/e$. Note that $n_\alpha \leq pe$, with equality iff $\alpha = e$. We put $\varepsilon_k = 0$ if $p = 1$, and $\varepsilon_k = 1$ else. With this notation at hand, we can write $S_A^{(\alpha)}(T)$ as

$$T^\alpha \left(\sum_{q \in \mathbb{N}} (qe + \alpha)^{t_A(\alpha)} (\mathbb{L}^{c(A)} T)^{qe} \right) - \varepsilon_k \mathbb{L}^{q_\alpha c(A)e} T^{n_\alpha} \left(\sum_{r \in \mathbb{N}} (epr + n_\alpha)^{t_A(\alpha)} (\mathbb{L}^{c(A)} T)^{epr} \right)$$

From [16, 6.2] and its proof, we can deduce that $S_A^{(\alpha)}(T)$ belongs to $\mathcal{M}^\alpha$, and that its specialization

$$P_k(S_A^{(\alpha)}(T)) \in \mathscr{F}[[T]]$$

has a pole of order $t_A(\alpha) + 1$ at $T = u^{-2c(A)}$, whose residue equals

$$(-1)^{t_A(\alpha)+1}(t_A(\alpha)!)u^{-2c(A)(\alpha + t_A(\alpha)+1)}(e^{-1} - \varepsilon_k(ep)^{-1})$$

This concludes the proof of Claim 1.

Now we prove claim 2. By our expression for $S_A^{(\alpha)}(T)$ and [16, 6.2], the rational function $S_A^{(\alpha)}(T)$ has degree < 0 if $t_A(\alpha) > 0$, so that we may assume that $t_A(\alpha) = 0$. In that case, we find

$$S_A^{(\alpha)}(T) = T^\alpha \left(\sum_{q \in \mathbb{N}} (\mathbb{L}^{c(A)}T)^{qe} \right) - \varepsilon_k \mathbb{L}^{q_\alpha c(A)e} T^{n_\alpha} \left(\sum_{r \in \mathbb{N}} (\mathbb{L}^{c(A)}T)^{epr} \right)$$

Direct computation shows that the degree of this rational function is ≤ 0, with equality iff $\alpha = e$ and $p = 1$. This concludes the proof. $\qquad\square$

8.4. **Elliptic curves.** As an example, we can give an explicit formula for the motivic zeta function of a tamely ramified elliptic K-curve E, in terms of the base change conductor $c(E)$. We refer to [13, 5.4.5] and [15, §8] for a table with the values of $c(E)$ (equivalently, the unique jump of E) for each of the Kodaira-Néron reduction types.

Proposition 8.6. *Let E be a tamely ramified elliptic curve over K, and denote by e the degree of the minimal extension of K where E acquires semi-abelian reduction. Denote by J the set of integers in $\{1, \ldots, ep-1\}$ that are prime to p and not divisible by e. In order to get uniformous formulas, we introduce an error factor ε_k which equals zero for $p = 1$ and which equals one for $p > 1$. We fix an algebraic closure $\mathbb{Q}^a$ of $\mathbb{Q}$, and denote by ξ_1 and ξ_2 the roots in $\mathbb{Q}^a$ of the characteristic polynomial $P_\sigma(t) \in \mathbb{Z}[t]$ of σ on $T_\ell A$.*

Then $e = \tau(c(E))$. If $c(E) = 0$ then $P_\sigma(t) = (t-1)^2$. If $c(E) = 1/2$ then $P_\sigma(t) = (t+1)^2$. If $c(E) \notin \{0, 1/2\}$ then $P_\sigma(t) = \Phi_{\tau(c(E))}(t)$.

Moreover, if we put

$$S_E(T) = \sum_{i \in J} (1 - (\xi_1)^i)(1 - (\xi_2)^i) \frac{\mathbb{L}^{1+\lfloor c(E)i \rfloor}T^i}{1 - \mathbb{L}^{c(E)ep}T^{ep}} \quad \in \mathcal{M}_k[[T]]$$

then

$$Z_E(T) = [B_E(e)] \cdot \left(\frac{\mathbb{L}^{c(E)e}T^e}{1 - \mathbb{L}^{c(E)e}T^e} - \varepsilon_k \frac{\mathbb{L}^{c(E)ep}T^{ep}}{1 - \mathbb{L}^{c(E)ep}T^{ep}} \right) + S_E(T)$$

if E has potential good reduction, and

$$Z_E(T) = \phi_E(e)(\mathbb{L}-1) \left(\frac{\mathbb{L}^{c(E)e}T^e}{(1 - \mathbb{L}^{c(E)e}T^e)^2} - \varepsilon_k \frac{p\mathbb{L}^{c(E)ep}T^{ep}}{(1 - \mathbb{L}^{c(E)ep}T^{ep})^2} \right) + S_E(T)$$

else.

Proof. The equality $e = \tau(c(E))$ and the expressions for $P_\sigma(t)$ follow immediately from Theorem 5.5. If d is an element of $\mathbb{N}'$ that is not divisible by e, then $E(d)$ has additive reduction, so that

$$\phi_E(d) = (1 - (\xi_1)^d)(1 - (\xi_2)^d)$$

by the trace formula in [25, 2.8]. This value only depends on the residue class of d modulo e, since ξ_1 and ξ_2 are e-th roots of unity.

If $d \in \mathbb{N}'$ is divisible by e, then $\phi_E(d) = 1$ if E has potential good reduction, and $\phi_E(d) = (d/e)\phi_E(e)$ if E has potential multiplicative reduction [16, 5.7]. Now the formulas for $Z_E(T)$ follow easily from the computation of $\mathrm{ord}_E(\cdot)$ in Proposition 7.5, and the expression for the motivic zeta function in Proposition 8.2. $\qquad\square$

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